

Gravity, Twistors and the MHV Formalism

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Abstract

We give a self-contained proof of the formula for the MHV amplitudes for gravity conjectured by Berends, Giele & Kuijf and use the associated twistor generating function to define a twistor action for the MHV diagram approach to gravity.

Starting from a background field calculation on a spacetime with anti self-dual curvature, we obtain a simple spacetime formula for the scattering of a single, positive helicity linearized graviton into one of negative helicity. Re-expressing our integral in terms of twistor data allows us to consider a spacetime that is asymptotic to a superposition of plane waves. Expanding these out perturbatively yields the gravitational MHV amplitudes of Berends, Giele & Kuijf.

We go on to take the twistor generating function off-shell at the perturbative level. Combining this with a twistor action for the anti self-dual background, the generating function provides the MHV vertices for the MHV diagram approach to perturbative gravity. We finish by extending these results to supergravity, in particular $\mathcal{N} = 4$ and $\mathcal{N} = 8$.

1 Introduction

Recent advances in understanding the perturbative structure of gravity (see *e.g.* [1–16]) have uncovered structures that are not visible in the standard spacetime formulation of general relativity. A particularly striking development has been the chiral MHV (Maximal Helicity Violating) diagram formulation [2, 13–15]. In this approach, the full perturbation theory for gravity, at least at tree level, is built up out of standard massless scalar propagators and *MHV vertices*. These vertices are off-shell continuations of amplitudes describing interactions of n linearized gravitons in momentum eigenstates, two of which have positive¹ helicity while $n - 2$ have negative helicity. Such amplitudes were first conjectured for Yang-Mills by Parke & Taylor [17] (and proved by Berends and Giele [?]) and

¹We will use Penrose conventions for twistor space, in which the amplitudes supported on a twistor line are ‘mostly minus’; these amplitudes are usually thought of as $\overline{\text{MHV}}$, but will be called MHV here. Our conventions are detailed at the end of the introduction.

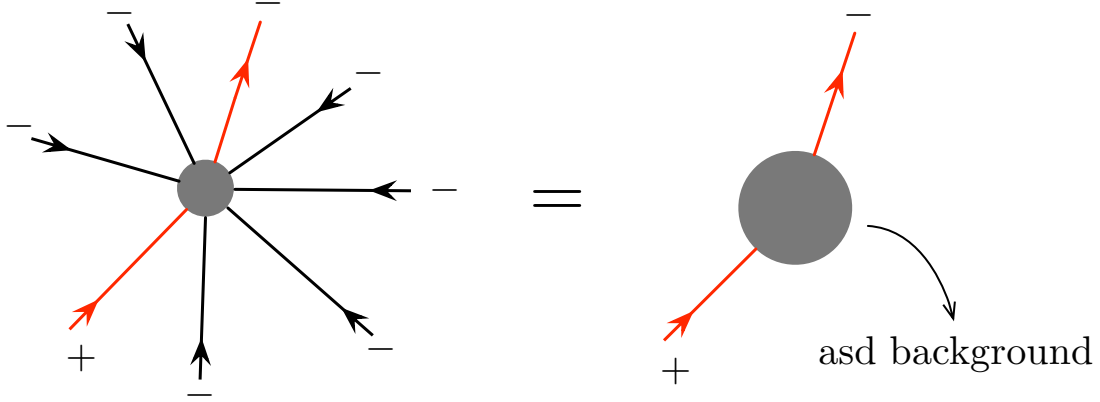


Figure 1: *Reversing the momentum of one of the positive helicity particles leads to the interpretation of the MHV amplitude as measuring the helicity-flip of a single particle which traverses a region of ASD background curvature.*

later a more complicated formula (95) for gravity was conjectured by Berends, Giele & Kuijf [18].

Both in gravity and Yang-Mills, MHV amplitudes are considerably simpler than a generic tree-level helicity amplitude. In particular, they may involve an arbitrary number of negative helicity gravitons (gluons) at little or no cost in complexity. Why should this be? Bearing in mind that a negative helicity graviton that has positive frequency is anti self-dual [19, 20], the picture in figure 1 interprets MHV amplitudes as measuring the helicity-flip of a single particle as it traverses a region of anti self-dual (ASD) background curvature. The asd Einstein equations, like the ASD Yang-Mills equations, have long been known to be completely integrable [21, 22] and lead to trivial scattering at tree-level. From this perspective, the key simplification of the MHV formalism arises because the ASD background, despite its non-linearities, can effectively be treated as a free theory. The MHV amplitudes themselves represent the first departure from anti self-duality.

The MHV formulation is essentially chiral. For gravity, this chirality suggests deep links to Plebanski's chiral action [23–25], to Ashtekar variables [25, 26] and to twistor theory [21, 27]. It is the purpose of this article to elucidate these connections further and to go some way towards a non-linear formulation that helps illuminate the underlying nonperturbative structure. Thus we begin in section 2 with a brief review of the Plebanski action, explaining how it can be used to expand gravity about its anti self-dual sector. Similar discussions have been given in [23, 28] and more recently [29] whose treatment we follow most closely.

On an ASD background, a linearized graviton has a canonically defined self-dual part, but its anti self-dual part shifts as it moves through the spacetime. We show in section 2.2 that the tree-level amplitude for this shift to occur is precisely measured by a simple space-time integral formula. This integral is a generating function for all the MHV amplitudes. To obtain them in their usual form, one must expand out the background field in terms of fluctuations around flat spacetime. Understanding how a non-linear anti self-dual field is composed of linearized gravitons is feasible precisely because the asd equations are inte-

grable, but nonetheless the inherent non-linearity makes this a rather complicated task on spacetime [30]. However, by going to twistor space and using Penrose’s non-linear graviton construction [21], the ASD background can be reformulated in an essentially linear way. Hence in section 3, after reviewing the relevant twistor theory of both linear gravity and non-linear ASD gravity, we obtain a twistor representation of the generating function using twistor integral formulæ for the spacetime fields. We will see that it is straightforward to construct a twistor space for a non-linear ASD spacetime that asymptotically is a linear superposition of momentum eigenstates. This uses a representation for the twistor space as an asymptotic twistor space constructed from the asymptotic data and is closely related to Newman’s \mathcal{H} -space construction [31]. Thus, we can use the twistor description to expand our generating function around Minkowski spacetime. A completely analogous story is true in Yang-Mills [32,33], with the corresponding twistor expression yielding all the Parke-Taylor amplitudes. This is reviewed in appendix B; some readers may find it helpful to refer to the (somewhat simpler) Yang-Mills case for orientation.

Performing the expansion, one finds that the n -point amplitude comes from an integral over the space of holomorphic twistor lines with n marked points. The marked points support operators representing the external gravitons; the 2 positive helicity gravitons are represented by 1-form insertions while the $n - 2$ negative helicity gravitons give insertions of vector fields. These vectors differentiate the external wavefunctions, leading to what is sometimes called ‘derivative of a δ -function support’. The 1-forms and vector fields really represent elements of certain cohomology classes on twistor space. It is interesting to note that these are the same cohomology groups that arise as (part of) the BRST cohomology in twistor-string theory [34,35], but here there are extra constraints which ensure that they represent Einstein, rather than conformal, gravitons. A string theory whose vertex operators satisfy these extra constraints was constructed in [36], although these models do not appear to reproduce the MHV amplitudes [37].

Integrating out the twistor variables finally yields the formula

$$\mathcal{M}^{(n)} = \frac{\kappa^{n-2}}{\hbar} \delta^{(4)} \left(\sum_{i=1}^n p_i \right) \times \left\{ \frac{[1n]^8}{[1n-1][n-1n][n1]} \frac{1}{C(n)} \prod_{k=2}^{n-1} \frac{\langle k | p_{k+1} + \cdots + p_{n-1} | n \rangle}{[kn]} + P_{\{2,\dots,n-2\}} \right\}, \quad (1)$$

for the n -particle amplitude $\mathcal{M}^{(n)}$, where $\kappa = \sqrt{16\pi G_N}$ and we have used the spinor-helicity formalism: the i^{th} external graviton is taken to have null momentum $p_{\alpha\dot{\alpha}}^{(i)} = |i\rangle[i|$, where $|i\rangle$ and $[i|$ respectively denote the anti-self-dual and self-dual spinor constituents of p_i , and $C(n)$ is the cyclic product $[12][23] \cdots [n-1n][n1]$. The symbol $P_{\{2,\dots,n-2\}}$ denotes a sum over permutations of gravitons 2 to $n-2$; the amplitude is completely symmetric in the external states (up to the overall factor $[1n]^8$ from the two positive helicity gravitons) once these permutations are accounted for. Equation (1) is not the original expression of BGK [18] and an analytic proof that the two forms coincide for arbitrary $n \geq 4$ is given in appendix A. The twistor formula also yields the correct 3-point amplitude, which is non-zero in complexified momentum space (although yields zero on a Lorentzian real slice). Our generating function may be simply extended to the case of MHV amplitudes in supergravity, and this is discussed in section 6 for $\mathcal{N} = 4$ and $\mathcal{N} = 8$ supergravity.

In the *MHV diagram formalism*, the full perturbation theory is reproduced from MHV amplitudes that are continued off-shell to provide vertices. These vertices are then connected together with propagators joining positive and negative helicity lines. With p such propagators, one obtains a N^p MHV amplitude, usually thought of in terms of the scattering of $2 + p$ positive helicity gravitons and an arbitrary number of negative helicity gravitons. In section 5 we continue our twistorial generating function off-shell and couple it to the twistor action for anti self-dual gravity constructed in [38]. The Feynman diagrams of the resulting action reproduce (in a certain gauge) the MHV diagram formalism for gravity. At present, we understand this action only in perturbation theory, and its validity as an action for gravity rests on the validity of the MHV diagram formalism. It would be very interesting to learn how the off-shell twistor action generates off-shell curved spacetime metrics, or to see if the existence of the twistor action implies that the MHV expansion is indeed valid.

The gravitational MHV amplitudes were originally calculated [18] using the Kawai, Llewelyn & Tye relations [41], and subsequently recalculated in a different form using the Britto, Cachazo, Feng & Witten recursion relations [42], suitably modified for gravity [5–7]. Although the BGK expression is strongly constrained by having the correct soft and collinear limits, strictly speaking, BGK were only able to prove that their formula followed from the KLT relations for $n \leq 11$ external particles. The formulæ obtained from BCFW recursion relations have also only been verified to be equivalent to the BGK expression up to this level. Our derivation is a complete constructive proof of the BGK formula (the formulæ of [5–7] are also independently proved). Evidence for a MHV diagram formulation of perturbative gravity has been discussed in [13–15], based on recursion relations. It has been established [15] that the MHV diagrams yield the correct n -graviton amplitudes, again for $n \leq 11$. Reference [15] also gives a generating function for MHV amplitudes in $\mathcal{N} = 8$ supergravity, taking the BGK amplitudes as an input.

Some steps towards an MHV action for gravity have been taken in [43], starting from lightcone gauge in spacetime and inspired by the work of Mansfield in Yang-Mills [44, 45]. A twistorial generating function which reproduces the gravity MHV amplitudes was constructed by Nair in [46]. Nair’s paper has influenced this one; the main difference is that we give an independent derivation of the amplitudes, starting from a spacetime formula for scattering off an ASD background. We also take a more geometrical perspective than [46]. A treatment of the MHV amplitudes that emphasizes their close connection to the integrability of asd backgrounds has been given in [30, 47] using ‘perturbiners’.

1.1 Conventions and notation

Flat Minkowski spacetime \mathbb{M} is taken to be \mathbb{R}^4 with metric of Lorentz signature $(+---)$ and with vector indices $a = 0, 1, 2, 3$. Let \mathbb{S}^+ and \mathbb{S}^- be the self-dual and anti self-dual spin spaces. Elements of \mathbb{S}^\pm will be taken to have dotted and undotted Greek indices respectively, *i.e.* $\dot{\alpha}, \dots = \dot{0}, \dot{1}$; $\alpha, \dots = 0, 1$. We denote the Levi-Civita alternating spinor by $\varepsilon_{\alpha\beta} = \varepsilon_{[\alpha\beta]}$, with $\varepsilon_{01} = -1$, *etc.* We often use the notation $r^\alpha \leftrightarrow |r\rangle$ and $s^{\dot{\alpha}} \leftrightarrow |s\rangle$ and then $[pr] = p^\alpha r^\beta \varepsilon_{\alpha\beta}$ and $\langle st \rangle = s^{\dot{\alpha}} t^{\dot{\beta}} \varepsilon_{\dot{\alpha}\dot{\beta}}$ denote the $SL(2, \mathbb{C})$ -invariant inner products. In complexified spacetime the two spin bundles will also be denoted \mathbb{S}^+ and \mathbb{S}^- . On a Lorentzian real slice they are related by complex conjugation $\overline{\mathbb{S}^+} = \mathbb{S}^-$, which therefore

exchanges dotted and undotted spinor indices. Vector indices $a = 0, 1, 2, 3$ can be replaced by spinor indices, so that the position vector of a point can be given as

$$x^{\alpha\dot{\alpha}} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 - x^3 \end{pmatrix}. \quad (2)$$

The Lorentz reality condition is $x^{\alpha\dot{\alpha}} = \bar{x}^{\dot{\alpha}\alpha}$, so that the rhs of (2) is a Hermitian matrix. We will often work on complexified spacetime, where x^a and $x^{\alpha\dot{\alpha}}$ are complex and the reality condition is dropped.

Projective twistor space \mathbb{PT}' is the space of totally null self-dual two planes (α -planes) in complexified spacetime. We describe \mathbb{PT}' using homogeneous coordinates $(\omega^\alpha, \pi_{\dot{\alpha}})$, with the incidence relation being $\omega^\alpha = ix^{\alpha\dot{\alpha}}\pi_{\dot{\alpha}}$; the solutions for $x^{\alpha\dot{\alpha}}$ holding $(\omega^\alpha, \pi_{\dot{\alpha}})$ constant defines the α -plane. In these conventions, an element of $H^1(\mathbb{PT}', \mathcal{O}(-2s-2))$ corresponds to an on-shell massless field of helicity s in spacetime by the Penrose transform. Thus a *negative* helicity gluon has homogeneity zero in twistor space, and the amplitudes supported on degree 1 holomorphic curve are ‘mostly minus’. We call such $\langle ++--\dots-- \rangle$ amplitudes MHV, although they are the complex conjugate of what is called an MHV amplitude in much of the scattering theory literature. With our conventions, Witten’s twistor-string theory [39] is really in dual twistor space. In Lorentzian signature, twistor space and its dual are related via complex conjugation, *i.e.* $(\omega^\alpha, \pi_{\dot{\alpha}}) \in \mathbb{PT}' \mapsto (\bar{\pi}_{\dot{\alpha}}, \bar{\omega}^{\dot{\alpha}}) \in \mathbb{PT}'^*$, reflecting the Lorentzian conjugation of Weyl spinors. For complexified spacetime, one often gives dual twistor space independent coordinates $(\lambda_\alpha, \mu^{\dot{\alpha}})$ which are the coordinates used in [39].

2 MHV Amplitudes on ASD Background Fields

2.1 The Plebanski action

The (complexified) spin group of a Lorentzian four manifold M is $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. Correspondingly, the tangent bundle TM decomposes into the self-dual and anti self-dual spin bundles \mathbb{S}^\pm as $TM \simeq \mathbb{S}^+ \otimes \mathbb{S}^-$. Each $SL(2, \mathbb{C})$ factor acts non-trivially on only either \mathbb{S}^+ or \mathbb{S}^- and so any connection on TM may be decomposed into connections on the two spin bundles as $\Gamma \oplus \tilde{\Gamma}$. Splitting the curvature two-form into its self-dual and anti self-dual parts R^\pm , one finds that $R^+ = R^+(\Gamma)$ and $R^- = R^-(\tilde{\Gamma})$ so that the self-dual (ASD) part of the curvature depends only on the connection on \mathbb{S}^+ (\mathbb{S}^-). (On a Lorentzian four-manifold $*^2 = -1$, so the SD/ASD curvatures are complex and $\bar{\Gamma} = \tilde{\Gamma}$, $R^+ = \bar{R}^-$. In Euclidean or split signature the spin connections and R^\pm are real and independent. We will mostly work on complexified spacetime, imposing reality conditions only at the end.)

Plebanski [23] gave a chiral action for Einstein’s general relativity that brings out this structure (see also [24, 25]). In his approach, the basic variables are the self-dual spin connection Γ , together with a tetrad of 1-forms $e^{\alpha\dot{\alpha}}$ which define the metric by

$$ds^2 = \varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}} e^{\alpha\dot{\alpha}} e^{\beta\dot{\beta}}, \quad (3)$$

where $\varepsilon_{\alpha\beta} = \varepsilon_{[\alpha\beta]}$, $\varepsilon_{01} = 1$ and similarly for $\varepsilon_{\dot{\alpha}\dot{\beta}}$. The components of the tetrad are defined by $e^{\alpha\dot{\alpha}} = e_a^{\alpha\dot{\alpha}} dx^a$ and form a vierbein. Plebanski’s action is a first-order theory in which

Γ and the tetrad are treated as independent *a priori*. In the absence of a cosmological constant, the action is

$$S[\Sigma, \Gamma] = \frac{1}{\kappa^2} \int_M \Sigma^{\dot{\alpha}\dot{\beta}} \wedge (d\Gamma + \Gamma \wedge \Gamma)_{\dot{\alpha}\dot{\beta}} \quad (4)$$

where $\kappa^2 = 16\pi G_N$ and $\Sigma^{\dot{\alpha}\dot{\beta}}$ are three self-dual two-forms, given in terms of the tetrad by $\Sigma^{\dot{\alpha}\dot{\beta}} = e^{\alpha(\dot{\alpha}} \wedge e_{\alpha}^{\dot{\beta})}$. It is a striking fact that $\tilde{\Gamma}$ plays no role in this action². It nevertheless describes full (non-chiral) Einstein gravity, as follows from the field equations

$$d\Sigma^{\dot{\alpha}\dot{\beta}} + 2\Gamma_{\dot{\gamma}}^{(\dot{\alpha}} \wedge \Sigma^{\dot{\beta})\dot{\gamma}} = 0 \quad (5)$$

$$\left(d\Gamma_{\dot{\alpha}\dot{\beta}} + \Gamma_{(\dot{\alpha}}^{\dot{\gamma}} \wedge \Gamma_{\dot{\beta})\dot{\gamma}} \right) \wedge e^{\alpha\dot{\alpha}} = 0. \quad (6)$$

The first of these is the condition that Γ is torsion-free, which fixes it in terms of the tetrad. Since (after an integration by parts) Γ appears in the action only algebraically, this equation may be viewed as a constraint. Imposing it in (6) implies that the Ricci curvature of the metric (3) vanishes, so that M satisfies the vacuum Einstein equations. Thus Plebanski's action is equivalent to the Einstein-Hilbert action (upto a topological term).

It is also possible to take $\Sigma^{\dot{\alpha}\dot{\beta}}$ to be an arbitrary set of self-dual 2-forms and view them as the basic variables, as was done in [24]. The condition that $\Sigma^{\dot{\alpha}\dot{\beta}}$ comes from a tetrad (*i.e.* $\Sigma^{\dot{\alpha}\dot{\beta}} = e^{\alpha(\dot{\alpha}} \wedge e_{\alpha}^{\dot{\beta})}$) is ensured by including a Lagrange multiplier to enforce $\Sigma^{\dot{\alpha}\dot{\beta}} \wedge \Sigma^{\dot{\gamma}\dot{\delta}} = 0$. In the present paper, this constraint will naturally be solved as part of the construction of $\Sigma^{\dot{\alpha}\dot{\beta}}$ from twistor space. We also remark that $\Sigma^{\dot{\alpha}\dot{\beta}}$ and $\Gamma_{\dot{\beta}}^{\dot{\alpha}}$ may be thought of as a 4-covariant form of Ashtekar variables [26]: if C is a spacelike Cauchy surface in M , then the restriction of $\Sigma^{\dot{\alpha}\dot{\beta}}$ to C gives Ashtekar's densitized triads³ $\sigma_{[jkl]}^{\dot{\alpha}\dot{\beta}i}$ via

$$\Sigma_{[ij]}^{\dot{\alpha}\dot{\beta}} \Big|_C = 3\sigma_{[ijk]}^{\dot{\alpha}\dot{\beta}k} \quad \Sigma_{[jk]}^{\dot{\alpha}\dot{\beta}} \delta^i_{\dot{l}} \Big|_C = \sigma_{[jkl]}^{\dot{\alpha}\dot{\beta}i}, \quad (7)$$

whereas the restriction of Γ to C is the Ashtekar-Sen-Witten connection (see [25] for details).

2.2 Linearizing around an anti self-dual background

We will be particularly interested in *anti self-dual* solutions to (5)-(6). On an ASD solution, the self-dual spin bundle $\mathbb{S}^+ \rightarrow M$ is flat, so Γ vanishes upto a gauge transform. The torsion-free constraint (5) becomes

$$d\Sigma^{\dot{\alpha}\dot{\beta}} = 0, \quad (8)$$

so that the *self-dual* part of the spin connection constructed from the tetrad $e^{\alpha\dot{\alpha}}$ must also be pure gauge. There are no constraints on the *anti self-dual* part of this connection, so

²Of course, one can still construct an ASD spin connection from the tetrad.

³ i, j, k, \dots are indices for the tangent space to C .

the associated Riemann tensor $R^a_{bcd}(e)$ need not vanish, but is purely asd. Decomposing a general Riemann tensor into irreducibles gives [48]

$$R_{abcd} = \Psi_{\alpha\beta\gamma\delta}\varepsilon_{\dot{\alpha}\dot{\beta}}\varepsilon_{\dot{\gamma}\dot{\delta}} + \tilde{\Psi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta} + \Phi_{\alpha\beta\dot{\gamma}\dot{\delta}}\varepsilon_{\dot{\alpha}\dot{\beta}}\varepsilon_{\gamma\delta} + \Phi_{\gamma\delta\dot{\alpha}\dot{\beta}}\varepsilon_{\alpha\beta}\varepsilon_{\dot{\gamma}\dot{\delta}} + \frac{R}{12}(\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}\varepsilon_{\dot{\alpha}\dot{\beta}}\varepsilon_{\dot{\gamma}\dot{\delta}} + \varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}\varepsilon_{\dot{\alpha}\dot{\gamma}}\varepsilon_{\dot{\beta}\dot{\delta}}) \quad (9)$$

where $\tilde{\Psi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} = \tilde{\Psi}_{(\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta})}$ and $\Phi_{\alpha\beta\dot{\gamma}\dot{\delta}} = \Phi_{(\dot{\alpha}\dot{\beta})(\gamma\delta)}$ are the spinor forms of the self-dual part of the Weyl tensor and the trace-free part of the Ricci tensor, respectively, and R is the scalar curvature. With vanishing cosmological constant, $R^a_{bcd}(e)$ is anti self-dual if and only if $\tilde{\Psi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}$, $\Phi_{\dot{\alpha}\dot{\beta}\gamma\delta}$ and R vanish. The ASD part $\Psi_{\alpha\beta\gamma\delta}$ of the Weyl tensor need not vanish (at least in complexified or Euclidean spacetime), but it obeys $\nabla^{\alpha\dot{\alpha}}\Psi_{\alpha\beta\gamma\delta} = 0$ as a consequence of the Bianchi identities on the ASD background. Anti self-dual spacetimes are sometimes known as ‘half-flat’ or ‘left-flat’. As discussed in [29], such left-flat spacetimes are all that survive in a chiral limit of the Plebanski action, obtained by rescaling $\Gamma \rightarrow \kappa^2\Gamma$ and then taking the limit $\kappa^2 \rightarrow 0$. In this chiral theory, Γ is independent of the tetrad even after the field equations are imposed.

In the full theory (4), set $\Sigma = \Sigma_0 + \sigma$ and $\Gamma = \Gamma_0 + \gamma$ to consider a small fluctuation on a background (Σ_0, Γ_0) . We will eventually take all fluctuations to be proportional to the coupling κ . When the background is anti self-dual (so Σ_0 is closed and Γ_0 vanishes), the fluctuations are subject to the linearized field equations

$$d\sigma^{\dot{\alpha}\dot{\beta}} = -2\gamma^{\dot{\alpha}}_{\dot{\gamma}}\Sigma_0^{\dot{\beta}\dot{\gamma}} \quad \text{and} \quad d\gamma_{\dot{\alpha}\dot{\beta}} \wedge e_0^{\beta\dot{\beta}} = 0. \quad (10)$$

Note that the exterior derivatives d here can be thought of as acting covariantly on the dotted spinor indices, since $\mathbb{S}^+ \rightarrow M$ is flat in the background. After some algebra, the second of these equations implies that

$$d\gamma_{\dot{\alpha}\dot{\beta}} = \tilde{\psi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}\Sigma_0^{\dot{\gamma}\dot{\delta}}, \quad (11)$$

where $\tilde{\psi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} = \tilde{\psi}_{(\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta})}$. Taking the exterior derivative of this equation and using (8) yields $\nabla^{\alpha\dot{\alpha}}\tilde{\psi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} = 0$, so $\tilde{\psi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}$ may be interpreted as a linearized self-dual Weyl tensor propagating on the asd background.

Since (10) are linearized, their space of solutions is a vector space V . If \mathcal{S} is the infinite dimensional space of solutions to the nonlinear field equations (5)-(6), then V may be thought of as the fibre of $T\mathcal{S}$ over the ASD background $(\Sigma_0, \Gamma_0) \in \mathcal{S}$. An on-shell linearized fluctuation (σ, γ) preserves the anti self-duality of the Riemann tensor if and only if it lies in a subspace $V^- \subset V$ defined by $\gamma^{\dot{\alpha}}_{\dot{\beta}} = 0$, modulo gauge. However, we cannot invariantly define an analogous subspace V^+ of self-dual solutions modulo gauge, because (*e.g.*) the condition that the variation of the ASD Weyl tensor should vanish is not true for infinitesimal diffeomorphisms and so such a definition is not gauge invariant. (In fact, it would be over-determined.) We can nevertheless define V^+ as the quotient $V^+ = V/V^-$ so that

$$V^+ = \{(\sigma, \gamma) \in V\} / \{(\sigma, \gamma) | \gamma^{\dot{\alpha}}_{\dot{\beta}} = d\mu^{\dot{\alpha}}_{\dot{\beta}}\} = \{\gamma | d\gamma_{\dot{\alpha}\dot{\beta}} \wedge e^{\alpha\dot{\alpha}} = 0\} / \{\gamma^{\dot{\alpha}}_{\dot{\beta}} = d\mu^{\dot{\alpha}}_{\dot{\beta}}\}. \quad (12)$$

An element $[\sigma, \gamma] \in V^+$ determines a unique non-zero linearized self-dual Weyl tensor by (11). The definitions of V^\pm are summarized in the exact sequence

$$0 \rightarrow V^- \rightarrow V \rightarrow V^+ \rightarrow 0, \quad (13)$$

where the second arrow is inclusion, and the third arrow is the map sending $(\sigma, \gamma) \rightarrow \gamma$ modulo linearized gauge transformations. Exactness means that if a linearized solution projects to zero in V^+ , then it necessarily comes from one in V^- . On a flat background, V decomposes as $V = V^+ \oplus V^-$, but on an ASD background such a global splitting is obstructed because elements of V^+ cannot globally be required to have non-vanishing anti self-dual parts. We will see that the MHV amplitudes precisely measure this obstruction.

2.3 Scattering of linearized fields

Figure 1 in the introduction realises the MHV amplitudes as the plane wave expansion of the amplitude for the scattering of a single, linearized graviton off an ASD background. The linearized graviton is taken to have positive helicity in the asymptotic past. To fix ideas, we consider a scattering process to take initial (characteristic) data from \mathcal{I}^- to data on \mathcal{I}^+ . Here, \mathcal{I}^\pm are future/past null infinity [48] and form the future/past boundaries of the conformal compactification of an asymptotically flat spacetime. They have the structure of lightcones (whose vertices are usually taken to be at infinity), so they have topology $S^2 \times \mathbb{R}$. In the conformal compactification of Minkowski space, the lightcone of a point on \mathcal{I}^- refocuses on a corresponding point of \mathcal{I}^+ and thus \mathcal{I}^\pm are canonically identified. The inversion $x^a \rightarrow x^a/x^2$ sends the lightcone of the origin to \mathcal{I}^\pm in the conformal compactification.

For our scattering process, the linearized graviton is prepared to have positive helicity on \mathcal{I}^- and scatters off the ASD background to emerge with negative helicity in the asymptotic future \mathcal{I}^+ . For *positive frequency* fields, states of positive or negative helicity are self-dual or anti self-dual, respectively [19,20]. On a curved spacetime, one can sometimes (perhaps with some gauge choices) define the positive/negative frequency splitting on an arbitrary Cauchy surface, but in general the results on different Cauchy surfaces will not agree, as is familiar *e.g.* from Hawking radiation. However, for an asymptotically flat spacetime, \mathcal{I}^\pm are lightcones at infinity' and have the same $S^2 \times \mathbb{R}$ topology as in Minkowski space. For these spacetimes, we can use Fourier analysis in the \mathbb{R} factors to perform the positive/negative frequency splitting at⁴ \mathcal{I}^+ or \mathcal{I}^- . Equivalently, one can split a field into parts that analytically continue into the upper and lower half planes respectively of the complexification \mathbb{C} of the \mathbb{R} generators. On an asymptotically flat spacetime that is *anti self-dual*, one can say more: as in Minkowski space, the lightcone emitted from an arbitrary point of \mathcal{I}^- refocusses at a point of \mathcal{I}^+ , so \mathcal{I}^\pm may again be canonically identified. (The reason for this will become transparent in the twistor formulation of the next section; essentially, identified points of \mathcal{I}^\pm correspond to the same Riemann sphere in twistor space.) Thus, on an ASD background, the positive/negative

⁴Strictly, to split into positive/negative frequency at \mathcal{I}^\pm , we must first perform a conformal rescaling so as to make sense of the limits of the fields at infinity. Such conformal rescalings can be canonically restricted to be constant along the generators [48] so there is no ambiguity in the splitting.

frequency splittings at \mathcal{I}^- and \mathcal{I}^+ agree, and it is easy to check they reproduce the standard splitting when the spacetime is flat. Thus we wish to find an expression for the scattering of a *self-dual* linearized graviton by an arbitrary asymptotically flat, asd spacetime M .

In the path integral approach, to compute the scattering amplitude, we formally consider the integral $\int [D\Sigma D\Gamma] e^{iS/\hbar}$, taken over all fields that approach the prescribed behaviour at \mathcal{I}^\pm . In the tree-level approximation, the path integral is given simply by evaluating $e^{iS/\hbar}$ on fields that extend this boundary configuration throughout the spacetime in accordance with the equations of motion, *i.e.* on $(\Sigma_0 + \sigma, \gamma)$. To leading order in the fluctuations, this is

$$e^{iS/\hbar} \approx 1 + \frac{i}{\kappa^2 \hbar} \int_M \Sigma_0^{\dot{\alpha}\dot{\beta}} \wedge \gamma_{\dot{\alpha}}^{\dot{\gamma}} \wedge \gamma_{\dot{\gamma}\dot{\beta}}. \quad (14)$$

The first term on the right hand side is the diagonal part of the S-matrix. The remaining part is the classical approximation to the transition amplitude we seek. This term is simply i/\hbar times the part of the Plebanski action that is lost in the chiral limit mentioned above. Indeed, because Γ satisfies $d\Gamma_{\dot{\alpha}\dot{\beta}} \wedge e^{\beta\dot{\beta}} = 0$ in the chiral theory, Γ is indistinguishable from the linearized fluctuation in Γ in the full theory. This field equation for γ also implies that the formula is gauge invariant since if we change $\gamma \rightarrow \gamma + d\chi$ with χ of compact support, the change in the integrand is clearly exact with compact support since $d(\gamma_{\dot{\alpha}\dot{\beta}} \wedge e^{\beta\dot{\beta}} \wedge e_{\dot{\beta}}^{\dot{\gamma}}) = 0$ and $d\Sigma_0^{\dot{\alpha}\dot{\beta}} = 0$.

In the MHV diagram formulation, the full classical theory can be built up from the complete set of MHV vertices, together with a propagator derived from the chiral theory⁵. Thus it is perhaps not surprising that all of the infinite number of MHV amplitudes should somehow be contained in this term. We will see later how to use this expression as a generating function for all the gravitational MHV amplitudes.

2.3.1 An alternative derivation

We will now rederive the expression for the scattering amplitude in more detail. Although this derivation is instructive, the impatient reader may prefer to skip ahead to the next section. Consider canonical quantization of the Plebanski action around an anti self-dual (rather than flat) background. The amplitude we seek might then be written as $\langle \Phi_{\text{out}} | \Phi_{\text{in}} \rangle_{\text{asd}}$, where $\langle \cdot | \cdot \rangle_{\text{asd}}$ is the inner product on the Hilbert space of the theory describing fluctuations around the asd background, and $\Phi_{\text{in}}, \Phi_{\text{out}}$ are in and out states of the appropriate helicity.

We can construct this inner product from the symplectic form on the phase space of the classical theory as follows (see *e.g.* [51, 52]). The space of solutions \mathcal{S} to (5)-(6) possesses a naturally defined closed two-form Ω defined using the boundary term in the variation of the action S . Letting δ denote the exterior derivative on the space of fields,

⁵As mentioned in the Introduction, the status of the MHV formalism in gravity - justified using recursion relations - requires a more complete understanding of the possible contribution from the ‘pole at infinity’ [15]. However, tree-level MHV diagrams in (super) Yang-Mills are known to be equivalent to Feynman diagrams [49, 50].

so that $\delta\Sigma^{\dot{\alpha}\dot{\beta}}$ and $\delta\Gamma_{\dot{\beta}}^{\dot{\alpha}}$ are one-forms on \mathcal{S} , Ω is given by

$$\Omega = \frac{1}{\kappa^2} \int_C \delta\Sigma^{\dot{\alpha}\dot{\beta}} \wedge \delta\Gamma_{\dot{\alpha}\dot{\beta}} \quad (15)$$

where C is a Cauchy surface in M . Ω is independent of the choice of Cauchy surface, because if C_1 and C_2 are two such surfaces bounding a region $D \subset M$ (*i.e.* $\partial D = C_1 - C_2$) then

$$\int_{C_1 - C_2} \delta\Sigma^{\dot{\alpha}\dot{\beta}} \wedge \delta\Gamma_{\dot{\alpha}\dot{\beta}} = \delta \int_{\partial D} \Sigma^{\dot{\alpha}\dot{\beta}} \wedge \delta\Gamma_{\dot{\alpha}\dot{\beta}} = \delta \int_D d \left(\Sigma^{\dot{\alpha}\dot{\beta}} \wedge \delta\Gamma_{\dot{\alpha}\dot{\beta}} \right). \quad (16)$$

Provided the field equations hold throughout D , this last term is $\delta^2 S$ and so vanishes because δ is nilpotent. Therefore, Ω is invariant under diffeomorphisms of M (whether or not these preserve C) and under rotations of the spin frame (it has no free dotted spinor indices). Moreover, Ω vanishes when evaluated on any changes in Σ and Γ that come from such a diffeomorphism or spin frame rotation, so it descends to a symplectic form on $\mathcal{S}/\text{Diff}_0^+(M)$. This symplectic form is real for real fields in Lorentzian signature. The quantum mechanical inner-product $\langle \cdot | \cdot \rangle$ is then defined as

$$\langle \cdot | \cdot \rangle = \frac{i}{\hbar} \Omega(\cdot, P_+ \cdot) \quad (17)$$

where P_+ projects states onto their positive frequency components⁶, defined at \mathcal{I}^\pm as above.

We can use the symplectic form to define a duality between V^+ and V^- . The symplectic form vanishes on restriction to the anti self-dual linearized solutions V^- (which have $\gamma = 0$, mod gauge). So, if $h_{a,b} = (\sigma_{a,b}, \gamma_{a,b})$ are two elements of $V \simeq T\mathcal{S}|_{M_{\text{asd}}}$ and $h_a \in V^- \subset V$, then $\Omega(\cdot, h_a)$ annihilates any part of h_b that is in V^- and we have

$$\Omega(h_b, h_a) = -\frac{1}{\kappa^2} \int_C \sigma_a^{\dot{\alpha}\dot{\beta}} \wedge \gamma_{b\dot{\alpha}\dot{\beta}} \quad (18)$$

for *any* $(\sigma_b, \gamma_b) \in V$. We see from this formula that the pairing only depends on γ_b , *i.e.* the projection of (σ_b, γ_b) into V^+ . Therefore, we have an isomorphism $V^+ \stackrel{\Omega}{\simeq} V^{-*}$.

We need to prepare our incoming field so that it is purely self-dual, so we need to construct a splitting of the sequence (13). This is easily done on \mathcal{I}^\pm using the standard expression of characteristic data for the gravitational field in terms of the asymptotic shear σ [48]. Since this expression may not be familiar to many readers, we give a somewhat formal, but equivalent definition: motivated by (18) we will say that a linearized field (σ_b, γ_b) is self-dual at \mathcal{I}^\pm if, given a one-parameter family C_t of Cauchy hypersurfaces, with $C_t \rightarrow \mathcal{I}^\pm$ as $t \rightarrow \pm\infty$, then

$$\lim_{t \rightarrow \pm\infty} \int_{C_t} \sigma_b^{\dot{\alpha}\dot{\beta}} \wedge \gamma_{c\dot{\alpha}\dot{\beta}} = 0, \quad \forall \gamma_c \in V^+. \quad (19)$$

⁶ P_+ is a choice of ‘polarization’ of the phase space in which positive/negative frequency states are taken to be holomorphic/antiholomorphic. We make this choice by defining it at null infinity, and no ambiguity arises as to whether future or past infinity is chosen in an asymptotically flat, ASD spacetime. One can check that (17) is positive definite, and linear/anti-linear in its left/right entries, with respect to the complex structure of the polarization.

We wish to consider the amplitude for a positive frequency, linearized solution h_1 that has positive helicity at \mathcal{I}^- to evolve into a positive frequency, negative helicity linearized solution at \mathcal{I}^+ by scattering off the ASD background. That is, h_1 is purely self-dual at \mathcal{I}^- so it satisfies (19), and we wish to know its anti self-dual part after evolving it to \mathcal{I}^+ . From the discussion above, we can extract this by computing the inner product with a linearized field h_2 that is purely self-dual (in V^{-*}) at \mathcal{I}^+ . Taking this inner-product at \mathcal{I}^+ , for positive frequency states the amplitude is

$$\langle h_2 | h_1 \rangle = \frac{i}{\hbar} \Omega(h_2, P_+ h_1) = -\frac{i}{\kappa^2 \hbar} \int_{\mathcal{I}^+} \sigma_1^{\dot{\alpha}\dot{\beta}} \wedge \gamma_{2\dot{\alpha}\dot{\beta}} \quad (20)$$

because (σ_2, γ_2) is purely self-dual at \mathcal{I}^+ . Now, $\partial M = \mathcal{I}^+ - \mathcal{I}^-$, so Stokes' theorem gives

$$\begin{aligned} \langle h_2 | h_1 \rangle &= -\frac{i}{\kappa^2 \hbar} \int_M \left(d\sigma_1^{\dot{\alpha}\dot{\beta}} \wedge \gamma_{2\dot{\alpha}\dot{\beta}} + \sigma_1^{\dot{\alpha}\dot{\beta}} \wedge d\gamma_{2\dot{\alpha}\dot{\beta}} \right) - \frac{i}{\kappa^2 \hbar} \int_{\mathcal{I}^-} \sigma_1^{\dot{\alpha}\dot{\beta}} \wedge \gamma_{2\dot{\alpha}\dot{\beta}} \\ &= \frac{i}{\kappa^2 \hbar} \int_M \Sigma_0^{\dot{\alpha}\dot{\beta}} \wedge \gamma_{1\dot{\alpha}}^{\dot{\gamma}} \wedge \gamma_{2\dot{\beta}\dot{\gamma}} - \sigma_1^{\dot{\alpha}\dot{\beta}} \wedge \tilde{\psi}_{2\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} \Sigma_0^{\dot{\gamma}\dot{\delta}} \\ &= \frac{i}{\kappa^2 \hbar} \int_M \Sigma_0^{\dot{\alpha}\dot{\beta}} \wedge \gamma_{1\dot{\alpha}}^{\dot{\gamma}} \wedge \gamma_{2\dot{\beta}\dot{\gamma}} . \end{aligned} \quad (21)$$

In going to the second line, we used the linearized field equations (10) together with the fact that $\int_{\mathcal{I}^-} \sigma_2^{\dot{\alpha}\dot{\beta}} \wedge \gamma_{1\dot{\alpha}\dot{\beta}} = 0$ because h_1 is purely self-dual at \mathcal{I}^- . The third line follows because $\sigma^{\dot{\alpha}\dot{\beta}} \wedge \Sigma_0^{\dot{\gamma}\dot{\delta}} = 0$ from the linearization of the constraint $\Sigma^{\dot{\alpha}\dot{\beta}} \wedge \Sigma^{\dot{\gamma}\dot{\delta}} = 0$ that ensures $\Sigma = \Sigma_0 + \sigma$ comes from a tetrad. Equation (21) agrees with the form of the tree amplitude computed before, as it should.

3 Twistor Theory for Gravity

Although we have argued that they are related, the expression (14) (or (21)) is still a far cry from the usual form of the MHV amplitudes, which live on a flat background spacetime. To connect the two pictures, we must expand out the ASD background in (21) in terms of plane wave perturbations away from Minkowski space. This background is explicitly present in (21) through $\Sigma_0^{\dot{\alpha}\dot{\beta}}$ and also implicit through the equations satisfied by the γ s. In order to perform the expansion we will have to use the integrability of the ASD interactions. Even so, constructing a fully nonlinear asd background that is asymptotically a superposition of negative helicity momentum eigenstates, and then using this background to evaluate (21) is a very complicated task. What enables us to proceed is the use of twistor theory, which brings out the integrability of the ASD sector and is therefore well-adapted to the problem at hand.

We now briefly review the twistor theory of linearized gravity on flat spacetime, before moving on to discuss Penrose's non-linear graviton construction [21] which gives the twistor description of an asd spacetime (see *e.g.* [48, 53, 54] for textbook treatments).

3.1 Linearized Gravity

We first review the basic twistor correspondence. The twistor space \mathbb{PT}' of flat spacetime is \mathbb{CP}^3 with a \mathbb{CP}^1 removed. We can describe \mathbb{CP}^3 using homogeneous coordinates $Z^I = (\omega^\alpha, \pi_{\dot{\alpha}})$ where $I = 0, \dots, 3$, while $\alpha = 0, 1$ and $\dot{\alpha} = \dot{0}, \dot{1}$ are spinor indices as before. In these coordinates, the line that is removed is given by $\pi_{\dot{\alpha}} = 0$, so that $\pi_{\dot{\alpha}} \neq 0$ on \mathbb{PT}' . Hence \mathbb{PT}' fibres over the \mathbb{CP}^1 whose homogeneous coordinates are $\pi_{\dot{\alpha}}$. Points $x \in \mathbb{C}^4$ of (complexified) spacetime with coordinates $x^{\alpha\dot{\alpha}}$ correspond to lines (\mathbb{CP}^1 s) in \mathbb{PT}' by the incidence relation

$$\omega^\alpha = ix^{\alpha\dot{\alpha}}\pi_{\dot{\alpha}}. \quad (22)$$

We will denote this line by L_x . The removed line $\pi_{\dot{\alpha}} = 0$ corresponds to a point at infinity in spacetime (the vertex of the lightcone at infinity).

We use the standard notation $\mathcal{O}(n)$ to denote the line bundle on \mathbb{CP}^m of Chern class n . Sections of $\mathcal{O}(n)$ can be identified with functions on the non-projective space of homogeneity degree n , so that $Z^I \partial f / \partial Z^I = n f$. We will use the same notation for line bundles over a projective line ($m = 1$) and over twistor space ($m = 3$).

The normal bundle to L_x in \mathbb{PT}' is $N_{L_x|\mathbb{PT}'} \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$. In particular, for $x = 0$, ω^α are coordinates along the fibres of the normal bundle to L_0 . Thus, in this flat case, \mathbb{PT}' is the total space of the normal bundle to a line. The incidence relation (22) identifies a point x with a holomorphic section $\mathbb{CP}^1 \rightarrow \mathbb{PT}'$ and the space of such sections $H^0(L_x, N_{L_x|\mathbb{PT}'}) \simeq \mathbb{C}^4$ is (complexified) flat spacetime.

The correspondence with flat spacetime can also be expressed in terms of the double fibration

$$\begin{array}{ccc} & P(\mathbb{S}^+) & \\ p \swarrow & & \searrow q \\ \mathbb{PT}' & & \mathbb{M} \end{array} \quad (23)$$

where $P(\mathbb{S}^+)$ is the projectivization of the bundle of dotted spinors, coordinatized by $(x^{\alpha\dot{\alpha}}, \pi_{\dot{\beta}})$ up to scaling of the π s, and $\mathbb{M} \simeq \mathbb{C}^4$ is complexified Minkowski space. The bundle $P(\mathbb{S}^+) \rightarrow \mathbb{M}$ is necessarily trivial, and the fibres $q^{-1}(x)$ are \mathbb{CP}^1 s coordinatized by $\pi_{\dot{\alpha}}$. Conversely, the fibres $p^{-1}(\omega^\alpha, \pi_{\dot{\alpha}})$ are the set of points $(x^{\alpha\dot{\alpha}}, \pi_{\dot{\alpha}})$ such that $\omega^\alpha = ix^{\alpha\dot{\alpha}}\pi_{\dot{\alpha}}$; given one such point (x_0, π) , this is the totally null, complex two-plane $x_0^{\alpha\dot{\alpha}} + \lambda^\alpha \pi^{\dot{\alpha}}$.

The Penrose transform represents linearized gravitons of helicities -2 and $+2$ on spacetime as elements of the twistor space cohomology groups $H^1(\mathbb{PT}', \mathcal{O}(2))$ and $H^1(\mathbb{PT}', \mathcal{O}(-6))$, respectively. In a Dolbeault framework, these are described locally by $(0, 1)$ -forms $h(Z)$ and $\tilde{h}(Z)$, homogeneous of degrees 2 and -6 . h and \tilde{h} thus obey $\bar{\partial}h = 0 = \bar{\partial}\tilde{h}$ and are defined up to the gauge freedom $h \sim h + \bar{\partial}\chi$, $\tilde{h} \sim \tilde{h} + \bar{\partial}\lambda$. We will suppress $(0, p)$ -form indices in what follows (and some readers may prefer to think in terms of a Čech picture of cohomology). The Penrose transforms of h and \tilde{h} are

$$\begin{aligned} \psi_{\alpha\beta\gamma\delta}(x) &= \int_{L_x} [\pi d\pi] \wedge p^* \left(\frac{\partial^4 h}{\partial \omega^\alpha \partial \omega^\beta \partial \omega^\gamma \partial \omega^\delta} \right) \\ \tilde{\psi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}(x) &= \int_{L_x} [\pi d\pi] \wedge \pi_{\dot{\alpha}} \pi_{\dot{\beta}} \pi_{\dot{\gamma}} \pi_{\dot{\delta}} p^* (\tilde{h}) \end{aligned} \quad (24)$$

where the pullback p^* simply imposes the incidence relation (22). Differentiating under the integral sign shows that ψ and $\tilde{\psi}$ obey the usual spin-2 (*i.e.* linearized Einstein) equations $\partial^{\alpha\dot{\alpha}}\psi_{\alpha\beta\gamma\delta} = 0$, $\partial^{\alpha\dot{\alpha}}\tilde{\psi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} = 0$ provided only that h and \tilde{h} are $\bar{\partial}$ -closed.

The cohomology class h plays an active role through its associated Hamiltonian vector field

$$V := I(\mathrm{d}h, \cdot) = I^{JK} \frac{\partial h}{\partial Z^J} \frac{\partial}{\partial Z^K} . \quad (25)$$

Here I is a holomorphic Poisson bivector of homogeneity -2 . It is determined by the line that was removed from \mathbb{CP}^3 to reach \mathbb{PT}' and has components

$$I^{JK} = \begin{pmatrix} \varepsilon^{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{so that} \quad I = \varepsilon^{\alpha\beta} \frac{\partial}{\partial \omega^\alpha} \wedge \frac{\partial}{\partial \omega^\beta} . \quad (26)$$

It follows that V in (25) represents an element of $H^1(\mathbb{PT}', T_{\mathbb{PT}'})$ and so describes a linearized complex structure deformation. We will study these deformations further in the next subsection.

A positive helicity graviton may also be represented by an element

$$B \in H^1(\mathbb{PT}', \Omega^{1,0} \otimes \mathcal{O}(-4)) \quad (27)$$

if, as well as having the standard gauge freedom $B \rightarrow B + \bar{\partial}\chi$ of a cohomology class, B is also subject to the additional gauge freedom

$$B \rightarrow B + \partial m + n[\pi \, \mathrm{d}\pi] . \quad (28)$$

Here, m and n are $(0, 1)$ -forms of homogeneity -4 and -6 respectively, while χ is a $(1, 0)$ -form of weight -4 . The freedom to add on arbitrary multiples of $[\pi \, \mathrm{d}\pi]$ means that only the part $B_\alpha \mathrm{d}\omega^\alpha$ of B along the fibres of $\mathbb{PT}' \rightarrow \mathbb{CP}^1$ contains physical information; the remaining freedom $B \rightarrow B + \partial m$ means that this physical information is captured by

$$I(\mathrm{d}B) = I^{IJ} \partial_I B_J = \varepsilon^{\alpha\beta} \frac{\partial B_\beta}{\partial \omega^\alpha} . \quad (29)$$

$I(\mathrm{d}B)$ is again in $H^1(\mathbb{PT}', \mathcal{O}(-6))$ and so can be identified with \tilde{h} . The Penrose transform of B is

$$\gamma_{\dot{\beta}}^{\dot{\alpha}} = 2 \int_{L_x} [\pi \, \mathrm{d}\pi] \wedge \pi^{\dot{\alpha}} \pi_{\dot{\beta}} p^*(B) \quad (30)$$

which, as our notation suggests, may be interpreted as a linearized self-dual spin connection. (The factor of 2 is for later convenience.) To see this, note first that (30) respects the gauge freedom (28) because any piece of p^*B proportional to $[\pi \, \mathrm{d}\pi]$ wedges to zero in (30), while adding on a total derivative $B \rightarrow B + \mathrm{d}m$ corresponds to the linearized gauge freedom $\gamma_{\dot{\beta}}^{\dot{\alpha}} \rightarrow \gamma_{\dot{\beta}}^{\dot{\alpha}} + \mathrm{d}\mu_{\dot{\beta}}^{\dot{\alpha}}$ of a spacetime connection. ($\mu_{\dot{\beta}}^{\dot{\alpha}}$ is the Penrose transform of m and satisfies the asd Maxwell equation $\partial_{\alpha\dot{\alpha}}\mu_{\dot{\beta}}^{\dot{\alpha}} = 0$.) The linearized spin connection

generates a linearized curvature fluctuation as it ought, since

$$\begin{aligned}
d\gamma_{\dot{\alpha}\dot{\beta}} &= 2dx^{\delta\dot{\delta}} \frac{\partial}{\partial x^{\delta\dot{\delta}}} \left(\int_{L_x} [\pi d\pi] \wedge \pi_{\dot{\alpha}} \pi_{\dot{\beta}} p^*(B) \right) \\
&= 2dx^{\delta\dot{\delta}} \wedge dx^{\gamma\dot{\gamma}} \int_{L_x} [\pi d\pi] \wedge \pi_{\dot{\alpha}} \pi_{\dot{\beta}} \pi_{\dot{\gamma}} \pi_{\dot{\delta}} p^* \left(\frac{\partial B_{\gamma}}{\partial \omega^{\delta}} \right) \\
&= dx^{\delta\dot{\delta}} \wedge dx_{\delta}^{\dot{\gamma}} \int_{L_x} [\pi d\pi] \wedge \pi_{\dot{\alpha}} \pi_{\dot{\beta}} \pi_{\dot{\gamma}} \pi_{\dot{\delta}} p^* (\tilde{h}) \\
&= \tilde{\psi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} dx^{\delta\dot{\delta}} \wedge dx_{\delta}^{\dot{\gamma}}
\end{aligned} \tag{31}$$

where in the second line we used the fact that $p^*B = iB_{\gamma}(ix \cdot \pi, \pi) dx^{\gamma\dot{\gamma}} \pi_{\dot{\gamma}} \pmod{[\pi d\pi]}$, which depends on x only through $\omega^{\gamma} = ix^{\gamma\dot{\gamma}} \pi_{\dot{\gamma}}$.

Plane wave gravitons (linearized spin-2 fields) of momentum $p_{\alpha\dot{\alpha}} = \tilde{k}_{\alpha} k_{\dot{\alpha}}$ may be described by twistor functions

$$h(Z) = \kappa \bar{\delta}_{(2)}([\pi k]) \exp(\langle \omega \tilde{k} \rangle) \quad \tilde{h}(Z) = \kappa \bar{\delta}_{(-6)}([\pi k]) \exp(\langle \omega \tilde{k} \rangle) . \tag{32}$$

where, for later use, we have taken all fluctuations to be proportional to the coupling $\kappa = \sqrt{16\pi G_N}$ and we follow [40, 57] in defining

$$\bar{\delta}_{(r)}([\pi k]) := \left(\frac{[\pi \alpha]}{[k \alpha]} \right)^{r+1} \bar{\partial} \frac{1}{[\pi k]} . \tag{33}$$

In this definition, $[\alpha]$ is a fixed dotted spinor introduced so that the δ -function (0,1)-forms $\bar{\delta}_{(r)}([\pi k])$ have homogeneity r in $[\pi]$. On the support of the $\bar{\delta}$ -function, $\pi_{\dot{\alpha}} \propto k_{\dot{\alpha}}$ so the momentum eigenstates (32) are in fact independent of the choice of $[\alpha]$. Note that, because of the weight of the $\bar{\delta}$ -function, h has weight -4 in the momentum spinor $[k]$ (counting $[\tilde{k}]$ as weight -1), while \tilde{h} has weight $+4$. This is as expected for states of helicity -2 and $+2$, respectively.

Likewise, the one-forms B may be taken to be

$$B(Z) = \kappa \frac{\langle \tilde{\beta} d\omega \rangle}{\langle \tilde{\beta} \tilde{k} \rangle} \bar{\delta}_{(-5)}([\pi k]) \exp(\langle \omega \tilde{k} \rangle) , \tag{34}$$

where the constant undotted spinor $\langle \tilde{\beta} |$ arises from the gauge freedom (28) in the definition of B . The choice of $\langle \tilde{\beta} |$ is arbitrary provided $\langle \tilde{\beta} \tilde{k} \rangle \neq 0$ reflecting the gauge freedom (28). It is easy to check that $I(dB) = \tilde{h}$, with \tilde{h} as above in (32).

We remark in passing that V represents an element of $H^1(\mathbb{P}T', T_{\mathbb{P}T'})$ together with the extra requirement (25) that it be Hamiltonian with respect to I , while (incorporating the redundancy $B \rightarrow B + \bar{\partial}\chi + \partial m$) B represents an element of $H^1(\mathbb{P}T', \Omega_{\text{cl}}^2 \otimes \mathcal{O}(-4))$ where Ω_{cl}^2 is the sheaf of closed (2,0)-forms, together with the extra requirement that $n[\pi d\pi]$ be taken equivalent to zero. Without the Hamiltonian and $n[\pi d\pi] \sim 0$ conditions, these cohomology groups represent states in conformal gravity. The extra conditions eliminate half the conformal gravity spectrum, reducing it to Einstein gravity as

above. The cohomology groups $H^1(\mathbb{P}\mathbb{T}', T_{\mathbb{P}\mathbb{T}'})$ and $H^1(\mathbb{P}\mathbb{T}', \Omega_{\text{cl}}^2 \otimes \mathcal{O}(-4))$ (together with their $\mathcal{N} = 4$ completions) define vertex operators in the Witten, Berkovits or heterotic twistor-string theories [34, 35]. String theories that impose the extra conditions were constructed in [36], but these theories only seem to describe the asd interactions of Einstein (super)gravity [37].

3.2 The Non-Linear Graviton

Penrose's non-linear graviton construction [21] associates a deformed twistor space \mathcal{PT} to a spacetime M with anti self-dual (ASD) Weyl tensor. In this correspondence, the structure of M is encoded into the deformed complex structure of the twistor space. For ASD spacetimes that also obey the vacuum Einstein equations, the twistor space fibres over \mathbb{CP}^1 and admits an analogue of the Poisson structure I along the fibres. We can still describe such a \mathcal{PT} using homogeneous coordinates $(\omega^\alpha, \pi_{\dot{\alpha}})$, where $\pi_{\dot{\alpha}}$ are holomorphic coordinates that are homogenous coordinates for the \mathbb{CP}^1 base. As in $\mathbb{P}\mathbb{T}'$, ω^α parametrize the fibres of $\mathcal{PT} \rightarrow \mathbb{CP}^1$, but in general they will no longer be holomorphic coordinates throughout the deformed twistor space. As in flat space, M is reconstructed as the space of degree-1 holomorphically embedded \mathbb{CP}^1 s inside \mathcal{PT} . For some fixed $x \in M$, we will again denote the corresponding \mathbb{CP}^1 by L_x . Although it will no longer have all the properties of a 'straight line', the normal bundle $N_{L_x|\mathcal{PT}}$ will still be $\mathcal{O}(1) \oplus \mathcal{O}(1)$ (as it was in the flat case) so that $H^0(L_x, N_{L_x|\mathcal{PT}}) \simeq \mathbb{C}^4$, which is identified as the *tangent* space $TM|_x$. Just as spacetime is no longer an affine vector space, \mathcal{PT} is no longer isomorphic to the total space of $N_{L_x|\mathcal{PT}}$. The correspondence may again be interpreted in terms of a double fibration

$$\begin{array}{ccc} & P(\mathbb{S}^+) & \\ p \swarrow & & \searrow q \\ \mathcal{PT} & & M \end{array} \quad (35)$$

as in (23). For a half-flat spacetime M that is sufficiently close to flat spacetime \mathbb{M} , the spin bundle is the product $\mathbb{CP}^1 \times M$.

The complex structure on \mathcal{PT} may be described in terms of a finite deformation of the flat background $\bar{\partial}$ -operator:

$$\bar{\partial} \rightarrow \bar{\partial} + V = \bar{\partial} + I(dh, \cdot) \quad (36)$$

with $I(dh, \cdot)$ as in equation (25). Only allowing Hamiltonian deformations of the $\bar{\partial}$ -operator ensures that \mathcal{PT} also fibres over \mathbb{CP}^1 and has a holomorphic Poisson structure I' on the fibres. This will be essential in the construction of the spacetime metric below. The deformed $\bar{\partial}$ -operator defines an integrable almost complex structure if and only if the Nijenhuis tensor

$$N = (\bar{\partial} + V)^2 \in \Omega^{0,2}(\mathbb{P}\mathbb{T}', T_{\mathbb{P}\mathbb{T}'}) \quad (37)$$

vanishes. For Hamiltonian deformations (36), one finds [38] $N = 0$ if

$$\bar{\partial}h + \frac{1}{2}\{h, h\} = 0. \quad (38)$$

There is a ‘Poisson diffeomorphism’ freedom generated by Hamiltonians χ which are smooth functions of weight two, because changing

$$h \rightarrow h + \bar{\partial}\chi + \{h, \chi\} \quad (39)$$

does not alter the complex structure. The diffeomorphism freedom can be fixed by requiring h to be holomorphic in ω^α and proportional to $\langle \bar{\pi} d\bar{\pi} \rangle$, so that its $(0,1)$ -form is purely along the base of the fibration $\mathcal{PT} \rightarrow \mathbb{CP}^1$. Any such h automatically leads to a vanishing Nijenhuis tensor. This gauge condition is natural in a scattering theory context, being essentially the same condition as is utilised in Newman’s formulation of the nonlinear graviton [22, 31, 55]. In Newman’s formulation (which will not be emphasized here), the holomorphic lines L_x are obtained from lightcone cuts of (complexified) null infinity \mathbb{CS} and can thus be reconstructed simply from the asymptotic data of the spacetime M , while h is interpreted as an integral of the asymptotic shear (the asymptotic characteristic data of M). Requiring that h be holomorphic in ω^α and proportional to $\langle \bar{\pi} d\bar{\pi} \rangle$ does not completely fix the gauge freedom (39). In Newman’s picture, the remaining freedom is fixed by additionally requiring that h depends on ω^α only through $\langle \omega \bar{\pi} \rangle$. We will implicitly use ‘Newman gauge’ in what follows: in particular, the twistor representatives of momentum eigenstates introduced in equation (32) are adapted to Newman gauge.

As mentioned above, each point $x \in M$ corresponds to a holomorphically embedded \mathbb{CP}^1 denoted by L_x . The flat space incidence relation $\omega^\alpha = ix^{\alpha\dot{\alpha}}\pi_{\dot{\alpha}}$ must be generalized, because ω^α is no longer a globally holomorphic coordinate on \mathcal{PT} . We thus represent $L_x \subset \mathcal{PT}$ by the deformed incidence relation

$$\omega^\alpha = F^\alpha(x, \pi) \quad (40)$$

where F^α has homogeneity one in $\pi_{\dot{\alpha}}$. The condition that L_x be holomorphic with respect to the deformed complex structure (36) is

$$0 = (\bar{\partial} + V)(\omega^\alpha - F^\alpha(x, \pi))|_{L_x} = V^\alpha|_{L_x} - \bar{\partial}F^\alpha(x, \pi) , \quad (41)$$

so that we obtain the condition

$$\bar{\partial}F^\alpha(x, \pi) = V^\alpha(F^\alpha(x, \pi), \pi) . \quad (42)$$

The restriction of V^α to L_x means that we set $\omega^\alpha = F^\alpha(x, \pi)$ in V , so that (42) is a nonlinear differential equation for F^α . This generally makes it very difficult to find explicit expressions for the holomorphic curves. As in \mathbb{PT}' , for fixed x the curve $L_x \subset \mathcal{PT}$ defined by (42) is a section of the fibration $\mathcal{PT} \rightarrow \mathbb{CP}^1$, holomorphic with respect to the deformed complex structure, and has normal bundle $N_{L_x|\mathcal{PT}} \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$. The deformation theory of Kodaira & Spencer implies that the family of lines in \mathbb{PT}' survive small deformations of the complex structure and form a four parameter family. Thus there will be a four parameter space of solutions to the nonlinear equation (42) and it is this parameter space that we identify with M .

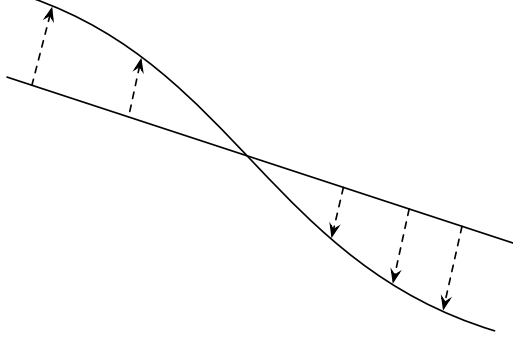


Figure 2: *Deformations of the complex structure induce deformations of the holomorphic curves. Identifying the four parameters x on which $F^\alpha(x, \pi)$ depends with spacetime coordinates, the normal vector $(F^\alpha - ix^{\alpha\dot{\alpha}}\pi_{\dot{\alpha}}) \partial/\partial\omega^\alpha$ on L_x connects the original twistor line $\omega^\alpha = ix^{\alpha\dot{\alpha}}\pi_{\dot{\alpha}}$ to the deformed curve.*

3.2.1 Constructing the spacetime metric

The space of degree one curves is naturally endowed with a conformal structure by requiring two points $x, y \in M$ to be connected by a null geodesic if their corresponding ‘lines’ $L_x, L_y \subset \mathcal{PT}$ intersect. Let us now show explicitly how to use the twistor data to construct a spacetime metric [21].

Consider the (weighted) 1-forms $[\pi d\pi]$ and $d\omega^\alpha - V^\alpha$. These forms are annihilated by contraction with the antiholomorphic vector fields of the deformed complex structure, and so define a basis of holomorphic forms on \mathcal{PT} . Note that the holomorphic form $[\pi d\pi]$ is unaltered compared to \mathbb{PT}^V ; this is a consequence of restricting to Hamiltonian complex structure deformations in (36). The holomorphic 3-form of weight +4 is therefore $\Omega_{\mathcal{PT}} = [\pi d\pi] \wedge (d\omega^\alpha - V^\alpha) \wedge (d\omega_\alpha - V_\alpha)$. Pulling back $\Omega_{\mathcal{PT}}$ to $P(\mathbb{S}^+)$ (*i.e.* imposing the incidence relation $\omega^\alpha = F^\alpha(x, \pi)$) gives

$$\begin{aligned} p^*\Omega_{\mathcal{PT}} &= [\pi d\pi] \wedge p^*(d\omega^\alpha - V^\alpha) \wedge p^*(d\omega_\alpha - V_\alpha) \\ &= [\pi d\pi] \wedge d_x F^\alpha \wedge d_x F_\alpha \end{aligned} \quad (43)$$

where d_x denotes the exterior derivative on $P(\mathbb{S}^+)$ holding $\pi_{\dot{\alpha}}$ constant, *i.e.* $d_x = dx^a \partial/\partial x^a$. (Possible terms in $d\bar{\pi}$ vanish by virtue of the holomorphy of these sections, while terms in $d\pi$ vanish by virtue of the fact that the expressions are wedged against $[\pi d\pi]$.) The requirement (42) that $L_x \subset \mathcal{PT}$ is a holomorphic line ensures $\bar{\partial}F^\alpha(x, \pi) = V^\alpha(F(x, \pi), \pi)$ so

$$\bar{\partial}(d_x F^\alpha \wedge d_x F_\alpha) = 2 d_x(\bar{\partial}F^\alpha) \wedge d_x F_\alpha = 2 \partial_\beta V^\alpha|_{L_x} d_x F^\beta \wedge d_x F_\alpha \quad (44)$$

where in the second term we used the fact that V^α depends on x only through $F^\beta(x, \pi)$. The wedge product implies this expression is antisymmetric in α, β and so in fact it vanishes because V is Hamiltonian. Therefore $d_x F^\alpha \wedge d_x F_\alpha$ is a two-form of homogeneity +2 in $\pi_{\dot{\alpha}}$ that is holomorphic along each \mathbb{CP}^1 . Consequently, by Liouville’s theorem,

$$p^*\Omega_{\mathcal{PT}} = -[\pi d\pi] \wedge q^*\Sigma^{\dot{\alpha}\dot{\beta}}(x)\pi_{\dot{\alpha}}\pi_{\dot{\beta}} \quad (45)$$

where $\Sigma^{\dot{\alpha}\dot{\beta}} \in \Omega^2(M, \text{Sym}^2 \mathbb{S}^+)$ are three spacetime two-forms, pulled back to $P(\mathbb{S}^+)$ by q^* . (The minus sign is for convenience.) We drop the pullback symbol q^* in what follows.

$\Sigma^{\dot{\alpha}\dot{\beta}}$ is automatically closed on spacetime, because $\Sigma^{\dot{\alpha}\dot{\beta}} \pi_{\dot{\alpha}} \pi_{\dot{\beta}} = d_x F^\alpha \wedge d_x F_\alpha$. The discussion around equation (8) then shows that the spacetime M is *necessarily anti self-dual*. Moreover, $\Sigma^{\dot{\alpha}\dot{\beta}}$ is *simple* by construction, so

$$\pi_{\dot{\alpha}} \pi_{\dot{\beta}} \Sigma^{\dot{\alpha}\dot{\beta}} = \pi_{\dot{\alpha}} e^{\alpha\dot{\alpha}} \wedge \pi_{\dot{\beta}} e_{\alpha}^{\dot{\beta}}. \quad (46)$$

for some tetrad $e^{\alpha\dot{\alpha}}$. This decomposition does not uniquely fix the tetrad: we can freely replace $e^{\alpha\dot{\alpha}}$ by $\Lambda^\alpha_\beta e^{\beta\dot{\alpha}}$ for $\Lambda^\alpha_\beta(x, \pi)$ an arbitrary element of $SL(2, \mathbb{C})$, as any such Λ^α_β drops out of equation (46). Comparing definitions shows that

$$\begin{aligned} p^*(d\omega^\alpha - V^\alpha) &= d_x F^\alpha \quad \text{mod } [\pi d\pi] \\ &= i \Lambda^\alpha_\beta e^{\beta\dot{\beta}} \pi_{\dot{\beta}} \quad \text{mod } [\pi d\pi]. \end{aligned} \quad (47)$$

Equations (46) & (47) generalize the flat spacetime formulæ⁷

$$\begin{aligned} p^*(d\omega^\alpha \wedge d\omega_\alpha) &= -dx^{\alpha\dot{\alpha}} \wedge dx_{\alpha}^{\dot{\beta}} \pi_{\dot{\alpha}} \pi_{\dot{\beta}} \quad \text{mod } [\pi d\pi] \\ p^*d\omega^\alpha &= i dx^{\alpha\dot{\alpha}} \pi_{\dot{\alpha}} \quad \text{mod } [\pi d\pi] \end{aligned} \quad (48)$$

arising from the incidence relation $\omega^\alpha = ix^{\alpha\dot{\alpha}} \pi_{\dot{\alpha}}$ in \mathbb{PT}' .

In (47), a choice of Λ^α_β fixes a choice of spin frame (for the undotted spinors) and hence a choice of tetrad $e^{\alpha\dot{\alpha}}$. However, although $\Lambda^\alpha_\beta(x, \pi)$ has weight zero in $\pi_{\dot{\alpha}}$, generically it is not π -independent. Because of this, it is not simply a local Lorentz transform on spacetime, but is best thought of as a holomorphic frame⁸ trivializing $N_{L_x|\mathcal{PT}} \otimes \mathcal{O}(-1)$ over L_x (see also [24, 25]). Note that since the normal bundle $N_{L_x|\mathcal{PT}} \simeq \mathcal{O}(1) \oplus \mathcal{O}(1)$, the bundle $N_{L_x|\mathcal{PT}} \otimes \mathcal{O}(-1)$ is indeed trivial on L_x . Its space of global holomorphic sections $H^0(L_x, \mathcal{O} \oplus \mathcal{O}) \simeq \mathbb{C}^2$ is precisely the fibre $\mathbb{S}^-|_x$ of the bundle of *anti* self-dual spinors on M .

4 Gravitational MHV amplitudes from twistor space

We now provide a twistorial description of $\langle h_2|h_1 \rangle$ by translating the right hand side of (21) using the Penrose integral transform. Finally, we will use the twistor description to expand around Minkowski space in plane waves, thus recovering the standard form of the MHV amplitudes. Underlying much of what follows is a presentation for the twistor data, going back to Newman [31], that relates directly to the asymptotic data at \mathcal{I} for the fields involved. By using sums of momentum eigenstates for the data at \mathcal{I} we guarantee that the fields and backgrounds that we work with are *asymptotically* superpositions of plane waves. Technically, ASD spacetimes constructed in this way are

⁷Strictly, equations (48) also includes a Λ^α_β in the definition of $p^*d\omega^\alpha$. Such a Λ relates the twistor coordinate index on ω^α to the undotted spacetime spinor index on $dx^{\alpha\dot{\alpha}}$. On a flat background these indices can be identified directly.

⁸ $\Lambda^\alpha_\beta(x, \pi)$ is thus somewhat analogous to the choice of holomorphic frame $H(x, \pi)$ that arises in a similar context for Yang-Mills, see equations (111)-(114).

not asymptotically flat along the directions of the plane waves. It is nevertheless possible to incorporate them into an asymptotically flat formalism at the expense of having to consider δ -function singularities in the asymptotic data (the asymptotic shear) as already apparent in (32) for the twistor representatives.

In section 2.2, the classical amplitude for a positive helicity graviton to cross an asymptotically flat ASD spacetime and emerge with negative helicity was shown to be

$$\langle h_n | h_1 \rangle = \frac{i}{\kappa^2 \hbar} \int_M \Sigma_0^{\dot{\alpha}\dot{\beta}} \wedge \gamma_n^{\dot{\gamma}}{}_{\dot{\alpha}} \wedge \gamma_1^{\dot{\beta}\dot{\gamma}} , \quad (49)$$

where $\Sigma_0^{\dot{\alpha}\dot{\beta}}$ is formed from the tetrad of the half-flat background and $\gamma_{1,n}$ are two linearized self-dual connections that are on-shell with respect to the linearized field equations (10). (The labelling 1, n is for later convenience.) We seek a twistorial interpretation of this term.

Firstly, the Penrose transform (30) of the linearized self-dual spin connection 1-form

$$\gamma_{\dot{\beta}}^{\dot{\alpha}} = 2 \int_{L_x} [\pi d\pi] \wedge \pi^{\dot{\alpha}} \pi_{\dot{\beta}} p^*(B) \quad (50)$$

also makes sense on an ASD background. To see this, first recall from section 2.2 that the background self-dual spin connection is flat on an ASD spacetime. It is therefore at most pure gauge and can be taken to vanish. The space of dotted spinors is then globally trivialized both on spacetime and on twistor space, so there is no difficulty in adding $\pi^{\dot{\alpha}} \pi_{\dot{\beta}}$ at different points of L_x in (50). As in equation (30) for flat space, the Penrose transform (50) is the pullback of the $(2,1)$ -form $[\pi d\pi] \wedge \pi^{\dot{\alpha}} \pi_{\dot{\beta}} B$ to $P(\mathbb{S}^+)$, pushed down to the ASD spacetime (*i.e.* integrated over the \mathbb{CP}^1 fibres of $P(\mathbb{S}^+) \xrightarrow{q} M$). To see that this pushdown is well-defined, note that for any vector field X on M , there is a unique vector field $\tilde{X} \in TP(\mathbb{S}^+)$ that obeys $\tilde{X} \lrcorner [\pi d\pi] = 0$ and whose projection $q_*(\tilde{X})$ to TM is again X . So for any such X , the integral $2 \int_{L_x} \tilde{X} \lrcorner ([\pi d\pi] \wedge \pi^{\dot{\alpha}} \pi_{\dot{\beta}} p^*B)$ is well-defined and equal to $X \lrcorner \gamma_{\dot{\beta}}^{\dot{\alpha}}$. Hence the integral (50) is also unambiguous. In particular, if $\{\nabla_{\gamma\dot{\gamma}}\}$ is a basis of TM dual to the tetrad, the components of the spin connection in this basis are given by contracting (50) with $\nabla_{\gamma\dot{\gamma}}$:

$$(\gamma_{\gamma\dot{\gamma}})^{\dot{\alpha}}{}_{\dot{\beta}} := \nabla_{\gamma\dot{\gamma}} \lrcorner \gamma_{\dot{\beta}}^{\dot{\alpha}} = 2 \int_{L_x} [\pi d\pi] \wedge \pi^{\dot{\alpha}} \pi_{\dot{\beta}} \pi_{\dot{\gamma}} B_{\alpha}(F, \pi) \Lambda^{\alpha}_{\gamma}(x, \pi) , \quad (51)$$

where we have used (47) to evaluate $\tilde{\nabla}_{\gamma\dot{\gamma}} \lrcorner p^*B$. The holomorphic frame Λ^{α}_{β} trivializes the anti self-dual spin bundle over L_x , thus allowing us to make sense of the integral of the indexed quantity⁹ B_{α} .

To construct the Penrose transform of the expression for $\langle h_n | h_1 \rangle$, we extract the components of each γ to obtain

$$\frac{i}{\kappa^2 \hbar} \int_M \Sigma_0^{\dot{\alpha}\dot{\beta}} \wedge \gamma_n^{\dot{\gamma}}{}_{\dot{\alpha}} \wedge \gamma_1^{\dot{\beta}\dot{\gamma}} = \frac{i}{2\kappa^2 \hbar} \int_M d\mu \gamma_n^{\gamma\dot{\gamma}\dot{\alpha}\dot{\beta}} \gamma_1^{\gamma\dot{\gamma}\dot{\alpha}\dot{\beta}} , \quad (52)$$

⁹A similar rôle is played by the holomorphic frame H in the Penrose transform of a background coupled self-dual Yang-Mills field, see (116).

where $d\mu := \Sigma_0^{\dot{\alpha}\dot{\beta}} \wedge \Sigma_{0\dot{\alpha}\dot{\beta}}$ is the volume form on M . Using the Penrose transform (51) in equation (49) gives

$$\langle h_n | h_1 \rangle = \frac{2i}{\kappa^2 \hbar} \int_{M \times \mathbb{CP}^1 \times \mathbb{CP}^1} d\mu [\pi_n d\pi_n][\pi_1 d\pi_1] B_{n\alpha}(F, \pi_n) \Lambda^{\alpha\gamma}(x, \pi_n) B_{1\beta}(F, \pi_1) \Lambda_\gamma^\beta(x, \pi_1) [\pi_n \pi_1]^3 \quad (53)$$

where $M \times \mathbb{CP}^1 \times \mathbb{CP}^1$ is the fibrewise product of $P(\mathbb{S}^+)$ with itself. The spinors $[\pi_1]$ and $[\pi_n]$ label to two copies of the \mathbb{CP}^1 fibres.

This formula currently describes the scattering of two positive helicity gravitons off a (fully non-linear) ASD background spacetime. In order to obtain the BGK amplitudes, we must expand the background spacetime M around Minkowski space \mathbb{M} . In principle, this can be done by iterating deformations of the twistor space caused by adding in negative helicity momentum eigenstates, and keeping track of the holomorphic degree-1 curves to construct the function $F^\alpha(x, \pi)$ explicitly (see [30, 56] for a discussion along these lines). In practice, constructing F^α in this way is complicated, and the difficulties are compounded by having to expand all the terms in (53). Instead, motivated by an analogous step at the same point in the Yang-Mills calculation (equation (118)), we seek a coordinate transformation of the spin bundle $P(\mathbb{S}^+) \rightarrow M$ that simplifies our task.

The desired coordinate transformation takes the form

$$(x^{\alpha\dot{\alpha}}, \pi_{\dot{\beta}}) \mapsto (y^{\alpha\dot{\alpha}}(x, \pi), \pi_{\dot{\beta}}) \quad \text{such that} \quad iy^{\alpha\dot{\alpha}}\pi_{\dot{\alpha}} = F^\alpha(x, \pi), \quad (54)$$

and may be viewed as a π -dependent coordinate transformation of M . Equation (54) replaces F^α by $iy^{\alpha\dot{\alpha}}\pi_{\dot{\alpha}}$, so that from the point of view of the (y, π) coordinates, we never need face the complicated problem of constructing $F^\alpha(x, \pi)$ explicitly! The price to be paid for this seemingly magical simplification is that generically, the \mathbb{CP}^1 fibres of $P(\mathbb{S}^+) \rightarrow M$ do not coincide with those of $P(\mathbb{S}^+) \rightarrow \mathbb{M}$ where here the y s are taken to be coordinates on \mathbb{M} ; in other words, the \mathbb{CP}^1 s of constant x (the twistor lines in \mathcal{PT}) are not the same as the \mathbb{CP}^1 s of constant y (the twistor lines in \mathbb{PT}'). There is some freedom in the definition of $y^{\alpha\dot{\alpha}}$ in (54). One natural choice that fits the bill is

$$y^{\alpha\dot{\alpha}}(x, \pi) = i \frac{F^\alpha(x, \xi)\pi^{\dot{\alpha}} - F^\alpha(x, \pi)\xi^{\dot{\alpha}}}{[\xi \pi]} \quad (55)$$

where $\xi^{\dot{\alpha}}$ is an arbitrary constant spinor. Note that if the background is actually flat, then $F^\alpha = ix^{\alpha\dot{\alpha}}\pi_{\dot{\alpha}}$ and we have simply $y^{\alpha\dot{\alpha}} = x^{\alpha\dot{\alpha}}$. Also note that the numerator vanishes at $[\pi] = [\xi]$, so the apparent singularity when $[\xi \pi] = 0$ is removable. Hence $y^{\alpha\dot{\alpha}}$ is smoothly (but not holomorphically) defined, and (y, π) are good coordinates on $P(\mathbb{S}^+)$, at least when the departure from flat spacetime is not too severe. Equation (55) explicitly shows that the \mathbb{CP}^1 s of constant x do not coincide with those of constant y , because y varies as we move along a \mathbb{CP}^1 fibre L_x .

We now pick a spacetime spin frame by requiring $\Lambda_\alpha^\beta(x, \xi) = \varepsilon_\alpha^\beta$. Then, using equation (47), the Jacobian of the coordinate transformation (55) with the spacetime tetrad $\nabla_{\alpha\dot{\alpha}}^{(x)}$ is found to be

$$\nabla_{\alpha\dot{\alpha}}^{(x)} y^{\beta\dot{\beta}} = \frac{1}{[\xi \pi]} \left(-i\Lambda_\alpha^\beta(x, \pi)\pi_{\dot{\alpha}}\xi^{\dot{\beta}} - \varepsilon_\alpha^\beta \xi_{\dot{\alpha}}\pi^{\dot{\beta}} \right). \quad (56)$$

This Jacobian has unit determinant because $\Lambda_\alpha^\beta \in SL(2, \mathbb{C})$, so $d\mu = d^4y \pmod{[\pi d\pi]}$. Furthermore, we see that

$$\pi^{\dot{\alpha}} \nabla_{\alpha \dot{\alpha}}^{(x)} = \pi^{\dot{\alpha}} \left(\nabla_{\alpha \dot{\alpha}}^{(x)} y^{\beta \dot{\beta}} \right) \frac{\partial}{\partial y^{\beta \dot{\beta}}} = \pi^{\dot{\alpha}} \frac{\partial}{\partial y^{\alpha \dot{\alpha}}} \quad (57)$$

which will be used in what follows.

We are not quite ready to put this coordinate transformation to use, because our expression (53) is written as an integral over the fibrewise product of the spin bundle with itself, rather than just as an integral over $P(\mathbb{S}^+)$. Since (54) does not map the fibres of $P(\mathbb{S}^+) \rightarrow M$ to the fibres of $P(\mathbb{S}^+) \rightarrow \mathbb{M}$, if the coordinate transformation is given by say $y(x, \pi_1)$, the π_n integral in equation (53) will not hold y constant for fixed (x, π_1) . We will deal with this by reformulating this second fibre integral as an inverse of the $\bar{\partial}$ -operator up the \mathbb{CP}^1 fibres of $P(\mathbb{S}^+)$ over M and perturbing about the the fibres of constant y .

We can understand $\bar{\partial}^{-1}$ as follows. First recall that on a single \mathbb{CP}^1 , any $(0,1)$ -form is automatically $\bar{\partial}$ -closed for dimensional reasons. The cohomology groups $H^{0,1}(\mathbb{CP}^1, \mathcal{O}(k))$ vanish for $k \geq -1$ and so any $(0,1)$ -form of homogeneity $k \geq -1$ is necessarily $\bar{\partial}$ -exact. Thus, if $\nu \in \Omega^{0,1}(\mathbb{CP}^1, \mathcal{O}(k))$ with $k \geq -1$ then $\bar{\partial}^{-1}\nu$ makes sense and is an element of $\Omega^0(\mathbb{CP}^1, \mathcal{O}(k))$. When $k \geq 0$, $\bar{\partial}^{-1}\nu$ is not uniquely defined because we can add to $\bar{\partial}^{-1}\nu$ a globally holomorphic function ρ of weight k , since $\bar{\partial}(\bar{\partial}^{-1}\nu + \rho) = \nu$. However, there are no global holomorphic functions of weight -1 , so when $k = -1$, $\bar{\partial}^{-1}\nu$ is unique. Explicitly, in terms of homogeneous coordinates $\pi_{\dot{\alpha}}$ on the \mathbb{CP}^1 , one takes¹⁰

$$\bar{\partial}_{21}^{-1}\nu := \frac{1}{2\pi i} \int_{\mathbb{CP}^1} \frac{[\pi_1 d\pi_1]}{[\pi_2 \pi_1]} \wedge \nu(\pi_1) , \quad (58)$$

which is indeed a 0-form of weight -1 in $\pi_2^{\dot{\alpha}}$. Taking $\bar{\partial}$ (with respect to $[\pi_2]$) of both sides shows that $\bar{\partial}\bar{\partial}^{-1} = 1$, because the only π_2 -dependence on the right is from the homogeneous form $1/2\pi i [\pi_2 \pi_1]$ of the standard Cauchy kernel for $\bar{\partial}^{-1}$. (In affine coordinates z on the Riemann sphere, $\pi = (1, z)$ and $[\pi_2 \pi_1] = z_1 - z_2$.)

To exploit this in our situation, first use (47) & (51) to rewrite (53) as

$$\langle h_n | h_1 \rangle = \frac{2i}{\kappa^2 \hbar} \int_{M \times \mathbb{CP}^1 \times \mathbb{CP}^1} d\mu [\pi_n d\pi_n] [\pi_1 d\pi_1] B_{n\alpha}(F, \pi_n) \Lambda^{\alpha\gamma}(x, \pi_n) \pi_n^{\dot{\gamma}} \tilde{\nabla}_{\gamma\dot{\gamma}} \lrcorner (B_1(F, \pi_1) [\pi_n \pi_1]^2) . \quad (59)$$

Next, note that $p^* B_1 \pi_1^{\dot{\alpha}} \pi_1^{\dot{\beta}} \pi_1^{\dot{\gamma}}$ has weight -1 in $[\pi_1]$ and is a $(0,1)$ -form on (the second) \mathbb{CP}^1 (valued also in $T^*M \otimes \text{Sym}^3 \mathbb{S}^+$). We then define

$${}_x \bar{\partial}_{n1}^{-1} (B \pi^{\dot{\alpha}} \pi^{\dot{\beta}} \pi^{\dot{\gamma}}) := \frac{1}{2\pi i} \int_{\mathbb{CP}^1} \frac{[\pi_1 d\pi_1]}{[\pi_n \pi_1]} \wedge B(F, \pi_1) \pi_1^{\dot{\alpha}} \pi_1^{\dot{\beta}} \pi_1^{\dot{\gamma}} \quad (60)$$

where the presubscript x emphasizes the fact that in this formula, $\bar{\partial}$ involves the $(0,1)$ -vector tangent to the \mathbb{CP}^1 fibres $q^{-1}(x)$. As above, this defines ${}_x \bar{\partial}^{-1}(B \pi^{\dot{\alpha}} \pi^{\dot{\beta}} \pi^{\dot{\gamma}})$ uniquely.

¹⁰The numerical prefactor $1/2\pi i$ of course involves the ratio of a circle's area to its radius, rather than the coordinate $\pi_{\dot{\alpha}}$.

Using this in equation (59) allows us to rewrite that equation as

$$\langle h_n | h_1 \rangle = -\frac{4\pi}{\kappa^2 \hbar} \int_{P(\mathbb{S}^+)} d\mu [\pi_n d\pi_n] \Lambda^{\alpha\beta} B_{n\beta}(F, \pi_n) \pi_n^{\dot{\alpha}} \tilde{\nabla}_{\alpha\dot{\alpha}} \lrcorner_x \bar{\partial}_{n1}^{-1} (B_1 [\pi_n \pi_1]^3) , \quad (61)$$

now interpreted as a (two-point) integral over the projective primed spin bundle.

We can now use the coordinate transformation to simplify the integral (61). Transforming to the (y, π) coordinates using equations (54) & (57) gives

$$\langle h_n | h_1 \rangle = -\frac{4\pi}{\kappa^2 \hbar} \int_{P(\mathbb{S}^+)} d^4 y [\pi_n d\pi_n] B_n^\alpha(y, \pi_n) \pi_n^{\dot{\alpha}} \frac{\partial}{\partial y^{\alpha\dot{\alpha}}} \lrcorner_x \bar{\partial}_{n1}^{-1} (B_1(y, \pi_1) [\pi_n \pi_1]^3) \quad (62)$$

now written as an integral on the spin bundle over flat spacetime. It remains to reformulate the operator ${}_x \bar{\partial}^{-1}$, the inverse of the $\bar{\partial}^{-1}$ operator on the \mathbb{CP}^1 s of constant x , in terms of ${}_y \bar{\partial}^{-1}$ the inverse of the $\bar{\partial}$ -operator on the \mathbb{CP}^1 s of constant y . These $\bar{\partial}$ -operators are essentially just the antiholomorphic tangent vector to the \mathbb{CP}^1 s of constant x or y and the relationship between them follows by the chain rule. Using equations (42) & (55) we find

$${}_x \bar{\partial} = {}_y \bar{\partial} + (\bar{\partial} y^{\alpha\dot{\alpha}}) \frac{\partial}{\partial y^{\alpha\dot{\alpha}}} = {}_y \bar{\partial} - i \frac{p^*(V^\alpha) \xi^{\dot{\alpha}}}{[\xi \pi]} \frac{\partial}{\partial y^{\alpha\dot{\alpha}}} , \quad (63)$$

where the extra term is the difference between an anti-holomorphic vector field tangent to the fibres of $P(\mathbb{S}^+) \rightarrow M$ and the anti-holomorphic vector field tangent to the fibres of $P(\mathbb{S}^+) \rightarrow \mathbb{M}$. Consequently, we see that¹¹ (somewhat formally)

$$\frac{1}{{}_x \bar{\partial}} = \frac{1}{{}_y \bar{\partial} + \mathcal{L}_{\tilde{V}}} , \quad (64)$$

where the right hand side of this equation involves the $\bar{\partial}$ -operator along the \mathbb{CP}^1 fibres in the (y, π) coordinates, together with the Lie derivative $\mathcal{L}_{\tilde{V}}$ along the vector field

$$\tilde{V} := -i \frac{p^*(V^\alpha) \xi^{\dot{\alpha}}}{[\xi \pi]} \frac{\partial}{\partial y^{\alpha\dot{\alpha}}} . \quad (65)$$

(We will often abuse notation by not distinguishing \tilde{V} from its pushdown $p_* \tilde{V} = V$ to twistor space.) The Lie derivative takes account of the fact that this operator acts on the form $B_\alpha dy^{\alpha\dot{\alpha}} \pi_{\dot{\alpha}}$; both the components B_α and basis forms $dy^{\alpha\dot{\alpha}}$ depend on y .

The operator $(\bar{\partial} + \mathcal{L}_V)^{-1}$ may be computed through its expansion

$$\frac{1}{\bar{\partial} + \mathcal{L}_V} = \frac{1}{\bar{\partial}} - \frac{1}{\bar{\partial}} \mathcal{L}_V \frac{1}{\bar{\partial}} + \frac{1}{\bar{\partial}} \mathcal{L}_V \frac{1}{\bar{\partial}} \mathcal{L}_V \frac{1}{\bar{\partial}} - \dots \quad (66)$$

where all the inverse $\bar{\partial}$ -operators now imply an integral over the \mathbb{CP}^1 s at constant $y^{\alpha\dot{\alpha}}$ (the holomorphic lines in \mathbb{PT}'). We have

$$\langle h_n | h_1 \rangle = \sum_{n=2}^{\infty} (-)^{n+1} \frac{4\pi}{\kappa^2 \hbar} \int d^4 y [\pi_n d\pi_n] B_n^\alpha \pi_n^{\dot{\alpha}} \frac{\partial}{\partial y^{\alpha\dot{\alpha}}} \lrcorner \left(\frac{1}{\bar{\partial}} \mathcal{L}_{V_{n-1}} \frac{1}{\bar{\partial}} \cdots \frac{1}{\bar{\partial}} \mathcal{L}_{V_2} \frac{1}{\bar{\partial}} B_1 [\pi_n, \pi_1]^3 \right) . \quad (67)$$

¹¹Equation (63) is analogous to the Yang-Mills equation $\mathcal{A} = -\bar{\partial} H H^{-1}$, while equation (64) is analogous to $H \bar{\partial}^{-1} H^{-1} = 1/(\bar{\partial} + \mathcal{A})$ used in appendix B.2.

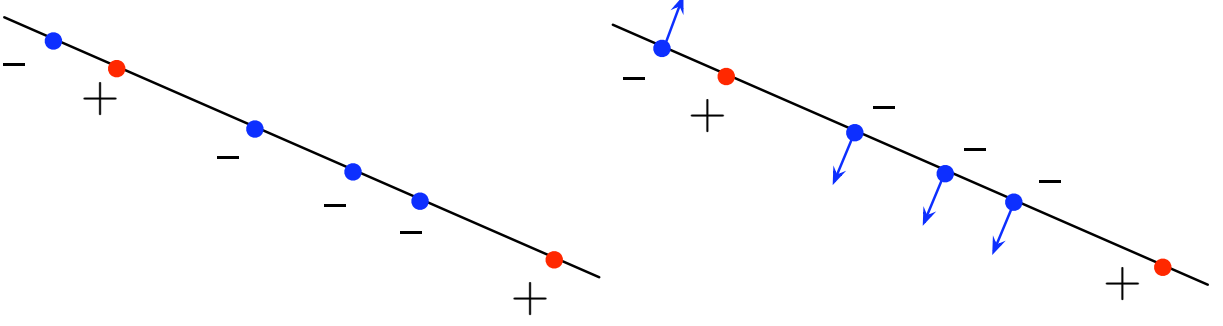


Figure 3: Yang-Mills (l) and gravitational (r) MHV amplitudes are supported on holomorphic lines in twistor space. For gravity, the negative helicity gravitons arise from insertions of normal vector fields, giving a perturbative description of the deformation of the line.

The inverse $\bar{\partial}$ -operators always act on sections of $\Omega^{0,1}(\mathbb{CP}^1, \mathcal{O}(-1) \otimes T^*M)$ and so are canonically defined as in equation (60), although here it is y rather than x that is being held constant. Because the vector fields \tilde{V} point in the y -direction, the Lie derivatives may be brought inside all the \mathbb{CP}^1 integrals, effectively commuting with the inverse $\bar{\partial}$ -operators. So the n^{th} -order term in the expansion is

$$\mathcal{M}_{\text{twistor}}^{(n)} := \frac{i^n}{(2\pi)^{n-2}} \frac{4\pi}{\kappa^2 \hbar} \int d^4y \prod_{i=1}^n \frac{[\pi_i d\pi_i]}{[\pi_{i+1} \pi_i]} B_n^\alpha \pi_n^{\dot{\alpha}} \frac{\partial}{\partial y^{\alpha \dot{\alpha}}} \lrcorner (\mathcal{L}_{V_{n-1}} \cdots \mathcal{L}_{V_2} B_1[\pi_n, \pi_1]^4) \quad (68)$$

where we have compensated for the fact that the integration measure $\prod_{i=1}^n [\pi_i d\pi_i]/[\pi_{i+1} \pi_i]$ includes an extra factor of $1/[\pi_n \pi_1]$ by increasing the power of $[\pi_n \pi_1]$ in the numerator. In this expression the n -point amplitude comes from an integral over the space of lines twistor space, with n insertions on the line, each of whose insertion point is integrated over. This is exactly the same picture as described in the appendix for Yang-Mills. For gravity, the $n-2$ vector fields differentiate the wavefunctions (as we will see explicitly later), leading to what is sometimes called ‘derivative of a δ -function’ support (see figure 3).

Our final task is to evaluate this expression when the external states are each the plane waves of (32)(34). From (25), the associated twistor space vector fields are $V(Z) = \kappa \bar{\delta}_{(1)}([\pi k]) e^{(\omega \tilde{k})} \tilde{k}^\alpha \partial / \partial \omega^\alpha$, so the vector fields on $P(\mathbb{S}^+)$ become

$$V(y, \pi) = -i\kappa \bar{\delta}_{(1)}([\pi k]) \exp(ip \cdot y) \frac{\tilde{k}^\alpha \xi^{\dot{\alpha}}}{[\xi \pi]} \frac{\partial}{\partial y^{\alpha \dot{\alpha}}} \quad (69)$$

using equation (65). Pulling the plane wave formula (34) for B back to $P(\mathbb{S}^+)$ gives

$$B = i\kappa \frac{\langle \tilde{\beta} | dy | \pi \rangle}{\langle \tilde{\beta} \tilde{k} \rangle} \bar{\delta}_{(-5)}([\pi k]) \exp(ip \cdot y) \quad (70)$$

in the (y, π) coordinates.

To evaluate (68), use the Cartan formula $\mathcal{L}_V = V \lrcorner d + dV \lrcorner$ to replace $\mathcal{L}_{V_{n-1}}$. The second term in Cartan's formula leads to a contribution

$$B_n^\alpha \pi_n^{\dot{\alpha}} \frac{\partial}{\partial y^{\alpha\dot{\alpha}}} \lrcorner d (V_{n-1} \lrcorner \mathcal{L}_{V_{n-2}} \cdots \mathcal{L}_{V_2} B_1) = B_n^\alpha \pi_n^{\dot{\alpha}} \frac{\partial}{\partial y^{\alpha\dot{\alpha}}} (V_{n-1} \lrcorner \mathcal{L}_{V_{n-2}} \cdots \mathcal{L}_{V_2} B_1) \quad (71)$$

to the integrand of (68). On the right hand side, $B_n^\alpha \pi_n^{\dot{\alpha}} \partial/\partial y^{\alpha\dot{\alpha}}$ simply differentiates the scalar $V_{n-1} \lrcorner \mathcal{L}_{V_{n-2}} \cdots \mathcal{L}_{V_2} B_1$. Because B_n is pulled back to $P(\mathbb{S}^+)$ from twistor space, it depends on y only through $y^{\alpha\dot{\alpha}} \pi_{\dot{\alpha}}$, so B_n^α may be brought inside the $\pi_n^{\dot{\alpha}} \partial/\partial y^{\alpha\dot{\alpha}}$ derivative. Hence (71) is a total derivative and may be discarded. Now, using the fact that $[d, \mathcal{L}_V] = 0$ for any vector field V , the remaining terms involve

$$\begin{aligned} B_n^\alpha \pi_n^{\dot{\alpha}} \frac{\partial}{\partial y^{\alpha\dot{\alpha}}} \lrcorner V_{n-1} \lrcorner (d\mathcal{L}_{V_{n-2}} \cdots \mathcal{L}_{V_2} B_1) \\ = B_n^\alpha \pi_n^{\dot{\alpha}} \frac{\partial}{\partial y^{\alpha\dot{\alpha}}} \lrcorner V_{n-1} \lrcorner \left(\mathcal{L}_{V_{n-2}} \cdots \mathcal{L}_{V_2} \left(\frac{\tilde{h}_1}{2} dy^{\gamma\dot{\gamma}} \wedge dy_{\gamma}^{\dot{\delta}} \pi_{1\dot{\gamma}} \pi_{1\dot{\delta}} \right) \right) \end{aligned} \quad (72)$$

where we have used $dB = (\tilde{h}/2) dy^{\alpha\dot{\alpha}} \wedge dy_{\alpha}^{\dot{\beta}} \pi_{\dot{\alpha}} \pi_{\dot{\beta}}$, which again follows because B is pulled back from a field on twistor space.

The key simplification that allows us to evaluate (72) comes from making the gauge choice $|\xi] = |n]$, where $|n]$ is the dotted momentum spinor of the positive helicity graviton represented by B_n . With this choice, the two-form $\mathcal{L}_{V_{n-2}} \cdots \mathcal{L}_{V_2} (\tilde{h}_1/2 dy^{\gamma\dot{\gamma}} \wedge dy_{\gamma}^{\dot{\delta}} \pi_{1\dot{\gamma}} \pi_{1\dot{\delta}})$ is contracted into the bi-vector $B_n^\alpha \pi_n^{\dot{\alpha}} V_{n-1}^\beta n^{\dot{\beta}} (\partial/\partial y^{\alpha\dot{\alpha}} \wedge \partial/\partial y^{\beta\dot{\beta}})$. But the momentum eigenstate B_n has support only when $|\pi_n] = |n]$, so this bi-vector is purely self-dual:

$$B_n^\alpha \pi_n^{\dot{\alpha}} \frac{V_{n-1}^\beta n^{\dot{\beta}}}{[n \pi_{n-1}]} \frac{\partial}{\partial y^{\alpha\dot{\alpha}}} \wedge \frac{\partial}{\partial y^{\beta\dot{\beta}}} = \frac{1}{2} \frac{\langle B_n V_{n-1} \rangle}{[n \pi_{n-1}]} \pi_n^{\dot{\alpha}} n^{\dot{\beta}} \frac{\partial}{\partial y^{\beta\dot{\beta}}} \wedge \frac{\partial}{\partial y^{\alpha\dot{\alpha}}} . \quad (73)$$

It is straightforward to check that because the vectors V_i are Hamiltonian, with our gauge choice, the bi-vector $\pi_n^{\dot{\alpha}} n^{\dot{\beta}} \partial/\partial y^{\alpha\dot{\alpha}} \wedge \partial/\partial y_{\alpha}^{\dot{\beta}}$ commutes with all the remaining Lie derivatives. Therefore, we may immediately contract this bivector with $dy^{\gamma\dot{\gamma}} \wedge dy_{\gamma}^{\dot{\delta}} \pi_{1\dot{\gamma}} \pi_{1\dot{\delta}}$ to obtain

$$\mathcal{M}_{\text{twistor}}^{(n)} = \frac{i^n}{(2\pi)^{n-2}} \frac{2\pi}{\kappa^2 \hbar} \int d^4 y \prod_{i=1}^n \frac{[\pi_i d\pi_i]}{[\pi_{i+1} \pi_i]} \frac{\langle B_n V_{n-1} \rangle}{[n \pi_{n-1}]} V_{n-2} \cdots V_2 (\tilde{h}_1) [\pi_n \pi_1]^5 [n \pi_1]^5 \quad (74)$$

where the remaining vector fields V_2 to V_{n-2} act simply by differentiating everything to their right.

To take account of the possible orderings of the external states, we insert

$$V_m = \kappa \sum_{i=2}^{n-1} \epsilon_i \bar{\delta}_{(1)}([\pi i]) e^{ip_i \cdot y} \frac{\tilde{i}^\alpha n^{\dot{\alpha}}}{[n \pi_m]} \frac{\partial}{\partial y^{\alpha\dot{\alpha}}} \quad (75)$$

for each vector field V_m at $(y^{\alpha\dot{\alpha}}, \pi_m^{\dot{\beta}})$, where the ϵ_i are expansion parameters labelling the physical external states. (We use the shorthand $p_i^{\alpha\dot{\alpha}} = \tilde{i}^\alpha i^{\dot{\alpha}}$.) Extracting the coefficient

of $\prod_{i=2}^{n-1} \epsilon_i$ and using the $\bar{\delta}$ -functions to integrate over the n insertion points gives the n -particle MHV amplitude as

$$\mathcal{M}_{\text{twistor}}^{(n)} = \frac{\kappa^{n-2}}{\hbar} \delta^{(4)} \left(\sum_{i=1}^n p_i \right) [1\,n]^8 \times \left\{ \frac{\langle \tilde{\beta}\,n-1 \rangle}{\langle \tilde{\beta}\,n \rangle [n-1\,n] [n\,1]^2} \frac{1}{C(n)} \prod_{k=2}^{n-2} \frac{\langle k|p_{k-1} + p_{k-2} + \cdots + p_2 + p_1|n \rangle}{[n\,k]} + P_{(2,\dots,n-1)} \right\}, \quad (76)$$

where $P_{(2,\dots,n-1)}$ is a sum over permutations of the vector fields. Consider the first (displayed) permutation. This is the same as the first term in $\mathcal{M}^{(n)}$ in equation (1), except for a factor

$$\frac{\langle \tilde{\beta}\,n-1 \rangle}{\langle \tilde{\beta}\,n \rangle [n\,1]} \times [1\,n-1] = -\frac{\langle \tilde{\beta}|p_{n-1}|1 \rangle}{\langle \tilde{\beta}|p_n|1 \rangle}. \quad (77)$$

This factor is independent of $2, \dots, n-2$, permuting the first term over gravitons 2 to $n-2$ will yield the same factor times the corresponding permutation of (1). Therefore we have

$$\mathcal{M}_{\text{twistor}}^{(n)} = -\frac{\langle \tilde{\beta}|p_{n-1}|1 \rangle}{\langle \tilde{\beta}|p_n|1 \rangle} \mathcal{M}^{(n)} + \text{other perms}. \quad (78)$$

The remaining permutations in (78) involve exchanging graviton $n-1$ with each of gravitons 2 to $n-2$. But since $\mathcal{M}^{(n)}$ is equal to the standard BGK amplitude (as proved in appendix A), we know (*e.g.* from Ward identities [59]) that it is in fact symmetric under exchange of any two like-helicity gravitons. Hence each term is proportional to $\mathcal{M}^{(n)}$ and we are left with an overall factor

$$-\sum_{i=2}^{n-1} \frac{\langle \tilde{\beta}|p_i|1 \rangle}{\langle \tilde{\beta}|p_n|1 \rangle} = -\frac{\langle \tilde{\beta}|p_2 + p_3 + \cdots + p_{n-1}|1 \rangle}{\langle \tilde{\beta}|p_n|1 \rangle} = 1. \quad (79)$$

Thus we have shown that (76) is really independent of $\tilde{\beta}$, and that $\mathcal{M}_{\text{twistor}}^{(n)} = \mathcal{M}^{(n)}$.

It is remarkable that the infinite series of n -particle MHV amplitudes may be constructed by expanding the square of the self-dual spin connection on an anti self-dual spacetime (49).

5 A Twistor Action for MHV Diagrams in Gravity

According to the MHV *diagram* formalism, initiated in [57] for Yang-Mills and [2,13,14] for gravity, one can recover the full perturbation theory by continuing the MHV amplitudes off-shell and connecting them together using propagators connecting positive and negative helicity lines¹². The MHV diagram formalism was first developed in the context of the ‘disconnected prescription’ of twistor-string theory [57], but soon after it was realized that one could also construct actions whose Feynman diagrams generate the Yang-Mills

¹²At the quantum level, this program works as stated only for supersymmetric theories [60].

MHV diagram formalism [32, 33, 40, 44, 45, 61, 62]. We now give a twistor action whose perturbation theory generates the MHV diagram formalism for gravity.

In section 3.2, ASD spacetimes were reformulated in terms of deformed twistor spaces by the nonlinear graviton construction [21]. The field equation on twistor space is the vanishing of the Nijenhuis tensor

$$N = I^{IJ} \partial_J \left(\bar{\partial} h + \frac{1}{2} \{h, h\} \right) \quad (80)$$

so that the almost complex structure $\bar{\partial} + I(dh, \cdot)$ is integrable and \mathcal{PT} is a complex threefold, obtained as a deformation of \mathbb{PT}' (see the discussion around equation (36)). In [38], a local twistor action whose field equations include the condition $N = 0$ was constructed. The action is written in terms of a field¹³ $h \in \Omega^{0,1}(\mathbb{PT}', \mathcal{O}(2))$ and $\tilde{h} \in \Omega^{0,1}(\mathbb{PT}', \mathcal{O}(-6))$, although we also here use $B \in \Omega^{1,1}(\mathbb{PT}', \mathcal{O}(-4))$. It takes a ‘BF’-like form

$$S = \int_{\mathbb{PT}'} \Omega \wedge I^{IJ} B_I \partial_J \left(\bar{\partial} h + \frac{1}{2} \{h, h\} \right) = - \int_{\mathbb{PT}'} \Omega \wedge \tilde{h} \left(\bar{\partial} h + \frac{1}{2} \{h, h\} \right) \quad (81)$$

where $\Omega = \epsilon_{IJKL} Z^I dZ^J \wedge dZ^K \wedge dZ^L / 4!$ is the canonical holomorphic 3-form of weight +4, I^{IJ} is the Poisson structure introduced in equation (26) and $\{\cdot, \cdot\}$ its associated Poisson bracket. Note that this Poisson bracket has weight -2 so that the action is well-defined on the projective space. In the first version, B plays the rôle of a Lagrange multiplier ensuring the vanishing of the Nijenhuis tensor. The second form follows upon integration by parts.

In the second form, the field equations of this action are

$$\bar{\partial} h + \frac{1}{2} \{h, h\} = 0 \quad \text{and} \quad \bar{\partial}_h \tilde{h} = 0, \quad \text{where} \quad \bar{\partial}_h f := \bar{\partial} f + \{h, f\}. \quad (82)$$

We also have the gauge freedom $h \rightarrow h + \bar{\partial}_h \chi$, $\tilde{h} \rightarrow \tilde{h} + \bar{\partial}_h \tilde{\chi}$. In the linearized theory, these imply that on-shell, h and \tilde{h} are representatives of the cohomology classes used to described linearized gravitons of helicities ± 2 in the Penrose transform, as reviewed in section 3.1. The gauge freedom may be fixed by using¹⁴ ‘CSW gauge’ [33, 57]: choose an antiholomorphic vector field $\bar{\eta}$ tangent to the fibres of $\mathbb{PT}' \rightarrow \mathbb{CP}^1$ and impose the axial gauge condition that $\bar{\eta} \lrcorner h = 0$ and $\bar{\eta} \lrcorner \tilde{h} = 0$. Imposing this gauge in (81), the cubic vertex vanishes and one is left with an off-diagonal kinetic term and a linear theory.

The other main ingredient in the MHV diagram formulation is the infinite set of MHV vertices: off-shell continuations of the MHV amplitudes. Using coordinates (y, π) for the spin bundle $P(\mathbb{S}^+) \rightarrow \mathbb{M}$ over Minkowski space, it follows from the previous section that in the twistor formulation these vertices arise from the expansion of

$$\int_{P(\mathbb{S}^+)} d^4 y \wedge [\pi_n d\pi_n] \wedge B^\alpha(y, \pi_n) \pi_n^{\dot{\alpha}} \frac{\partial}{\partial y^{\alpha\dot{\alpha}}} \lrcorner \left(\frac{1}{\bar{\partial} + \mathcal{L}_{\tilde{V}}} B(y, \pi_1) [\pi_n \pi_1]^3 \right) \quad (83)$$

¹³We abuse notation by not distinguishing the (0,1)-forms h, \tilde{h} from their cohomology classes.

¹⁴The Newman gauge of section 3.2 implies CSW gauge, but also enforces other conditions appropriate only when the fields are on-shell.

where we interpret B as the pullback to $P(\mathbb{S}^+)$ of an arbitrary element of $\Omega^{1,1}(\mathbb{P}\mathbb{T}', \mathcal{O}(-4))$ (*i.e.*, not necessarily obeying $\bar{\partial}B = 0$). Likewise, \tilde{V} is here interpreted as in (65):

$$\tilde{V} = -i \frac{p^*(V^\alpha) \xi^{\dot{\alpha}}}{[\xi \pi]} \frac{\partial}{\partial y^{\alpha \dot{\alpha}}} \quad \text{where} \quad V_\alpha = \frac{\partial h}{\partial \omega^\alpha}. \quad (84)$$

The inverse operator $1/(\bar{\partial} + \mathcal{L}_{\tilde{V}})$ is again understood through its infinite series expansion (66) leading to an infinite sequence of MHV vertices. These only involve the components of the (0,1)-forms B and h that are tangent to the \mathbb{CP}^1 base of $\mathbb{P}\mathbb{T}' \rightarrow \mathbb{CP}^1$.

The choice of the vector field $\bar{\eta}$ corresponds to the choice of spinor used by [57]. As described for Yang-Mills in [40], it enters into the definition of the propagator which gives the CSW rule for extending the MHV amplitudes off-shell. The discussion in [40] applies here directly with just a shift in homogeneities. Therefore, treating h and B as the fundamental fields, in CSW gauge, the Feynman diagrams of the action

$$S[B, h] = \int_{\mathbb{P}\mathbb{T}'} \Omega \wedge I^{IJ} B_I \partial_J \left(\bar{\partial} h + \frac{1}{2} \{h, h\} \right) + \int_{P(\mathbb{S}^+)} d^4 y \wedge [\pi_n d\pi_n] \wedge B^\alpha(y, \pi_n) \pi_n^{\dot{\alpha}} \frac{\partial}{\partial y^{\alpha \dot{\alpha}}} \lrcorner \left(\frac{1}{\bar{\partial} + \mathcal{L}_{\tilde{V}}} B(y, \pi_1) [\pi_n \pi_1]^3 \right) \quad (85)$$

reproduces the MHV diagram formulation of gravity.

6 Supergravity

Supertwistor space $\mathbb{P}\mathbb{T}'_{[\mathcal{N}]}$ is the projectivisation of $\mathbb{C}^{4|\mathcal{N}}$ where we have adjoined \mathcal{N} anti-commuting homogeneity degree 1 coordinates ψ^A , $A = 1, \dots, \mathcal{N}$. In Penrose conventions, the space of holomorphic lines in $\mathbb{P}\mathbb{T}'_{[\mathcal{N}]}$ is anti-chiral superspace $\mathbb{M}_{[\mathcal{N}]}$ with coordinates $(x^{\alpha \dot{\alpha}}, \tilde{\theta}^{A \dot{\alpha}})$, where $\tilde{\theta}^{A \dot{\alpha}}$ are anti-commuting. The flat space incidence relation (40) is augmented to

$$\omega^\alpha = ix^{\alpha \dot{\alpha}} \pi_{\dot{\alpha}} \quad \psi^A = \tilde{\theta}^{A \dot{\alpha}} \pi_{\dot{\alpha}}. \quad (86)$$

The linear Penrose transform of section 3.1 extends [63] to one between cohomology classes on $\mathbb{P}\mathbb{T}'_{[\mathcal{N}]}$ and superfields on $\mathbb{M}_{[\mathcal{N}]}$. In particular, h naturally extends to an (on-shell) superfield¹⁵ $\mathcal{H} \in H^1(\mathbb{P}\mathbb{T}'_{[\mathcal{N}]}, \mathcal{O}(2))$ that is holomorphic in ψ^i . That is, \mathcal{H} has component expansion

$$\mathcal{H}(Z, \psi) = h(Z) + \psi^A \lambda_A(Z) + \dots + (\psi^1 \psi^2 \dots \psi^{\mathcal{N}}) \phi(Z) \quad (87)$$

where the coefficient of $(\psi)^k$ may be represented by a (0,1)-form on the standard twistor space $\mathbb{P}\mathbb{T}'$ and has homogeneity $2 - k$. \mathcal{H} generates Poisson deformations of the complex structure of the twistor superspace through its associated Hamiltonian vector superfield $I(d\mathcal{H}, \cdot)$ [38]. As a superfield it represents on-shell spacetime fields of helicities $-2, -\frac{3}{2}, \dots, -2 + \frac{\mathcal{N}}{2}$. When $\mathcal{N} < 8$, the conjugate graviton supermultiplet is represented by a twistor superfield $\tilde{\mathcal{H}} \in H^1(\mathbb{P}\mathbb{T}'_{[\mathcal{N}]}, \mathcal{O}(\mathcal{N} - 6))$. As in the non-supersymmetric case, $\tilde{\mathcal{H}}$

¹⁵ $\mathbb{P}\mathbb{T}'_{[\mathcal{N}]}$ is a split supermanifold, whose cohomology is generated by that of the base.

may equivalently be represented by a superfield $\mathcal{B} \in H^1(\mathbb{PT}_{[\mathcal{N}]}, \Omega^1(\mathcal{N} - 4))$, modulo the gauge equivalence $\mathcal{B} \rightarrow \mathcal{B} + dm(Z, \psi) + n(Z, \psi)[\pi d\pi]$.

A particularly interesting case is $\mathcal{N} = 4$, for which twistor space is a Calabi-Yau supermanifold, *i.e.*, it admits a global holomorphic volume (integral) form. The Calabi-Yau property singles out $\mathcal{N} = 4$ twistor space as a natural target for a string theory [39]. When $\mathcal{N} = 4$, $\mathcal{B}(Z, \psi)$ has homogeneity zero and there is a natural extension of the action (85):

$$S_{\mathcal{N}=4} = \int_{\mathbb{PT}'_{[4]}} d^4\psi \Omega \wedge I^{IJ} \mathcal{B}_I \partial_J \left(\bar{\partial} \mathcal{H} + \frac{1}{2} \{ \mathcal{H}, \mathcal{H} \} \right) + \int_{P(\mathbb{S}_{[4]}^+)} d^{4|8} x \wedge [\pi_n d\pi_n] \wedge \frac{\mathcal{B}_n^\alpha \xi^{\dot{\alpha}}}{[\xi \pi_n]} \frac{\partial}{\partial x^{\alpha \dot{\alpha}}} \lrcorner \left(\frac{1}{\bar{\partial} + \mathcal{L}_{\mathcal{V}}} \mathcal{B} \right). \quad (88)$$

where in the second term, \mathcal{B} is pulled back to the superspace spin bundle $P(\mathbb{S}_{[4]}^+)$ and \mathcal{V} is the vector field on $P(\mathbb{S}_{[4]}^+)$ defined by (65) (or (84)) with V replaced by \mathcal{V} . In this $\mathcal{N} = 4$ formula, the inverse $\bar{\partial}$ -operators act on (0,1)-forms of vanishing weight. Although $\bar{\partial}^{-1}$ is not obstructed on such forms, it is ambiguous. The freedom can be fixed by adding a constant so that $\bar{\partial}_{21}^{-1} \mathcal{B}$ vanishes when $|\pi_2| = |\xi|$. With this choice,

$$\bar{\partial}_{21}^{-1} \mathcal{B}(x, \tilde{\theta}, \pi_2) := \frac{1}{2\pi i} \int_{\mathbb{CP}^1} \frac{[\pi_1 d\pi_1]}{[\pi_2 \pi_1]} \frac{[\xi \pi_2]}{[\xi \pi_1]} \mathcal{B}(x, \tilde{\theta}, \pi_1) \quad (89)$$

which has homogeneity zero in $|\pi_2|$ and satisfies $\bar{\partial} \bar{\partial}^{-1} \mathcal{B} = \mathcal{B}$; the integrand in the second term of (88) then has vanishing weight in each \mathbb{CP}^1 and is thus well-defined. It is easy to check that the truncation of (88) to $\mathcal{N} = 0$ reproduces (85). As in [64], when \mathcal{B} and \mathcal{V} are on-shell with respect to the local $\mathcal{N} = 4$ twistor action and are taken to be the twistor momentum eigenstates

$$\mathcal{V}(Z, \psi) = \kappa \bar{\delta}_{(1)}([\pi k]) \exp \left(\langle \omega \tilde{k} \rangle + \psi^A \zeta_A \right) \tilde{k}^\alpha \frac{\partial}{\partial \omega^\alpha} \quad (90)$$

$$\mathcal{B}(Z, \psi) = \kappa \bar{\delta}_{(-5)}([\pi k]) \exp \left(\langle \omega \tilde{k} \rangle + \psi^A \zeta_A \right) \frac{\langle \tilde{\beta} d\omega \rangle}{\langle \tilde{\beta} \tilde{k} \rangle}, \quad (91)$$

then the coefficients of the external Grassmann parameters ζ_A in an expansion of the non-local term give the MHV amplitudes for arbitrary external members of the $\mathcal{N} = 4$ supermultiplet.

Although $\mathcal{N} = 4$ twistor supersymmetry seems natural in twistor-string theory, $\mathcal{N} = 8$ supergravity is usually thought of as more fundamental. The $\mathcal{N} = 8$ graviton supermultiplet is CPT self-conjugate, and this fact has recently been argued to underlie many surprisingly simplifications in the S-matrix [9]. Thus, on twistor space, the complete multiplet is represented by a single superfield

$$\mathcal{H}(Z, \psi) = h(Z) + \psi^A \lambda_A(Z) + \cdots + (\psi)^8 \tilde{h}(Z) \quad (92)$$

In the case that the external states are on-shell momentum eigenstates, represented on twistor space by the Newman gauge expression (90), the MHV scattering of arbitrary

members of the $\mathcal{N} = 8$ multiplet is described by the formula

$$\mathcal{M}_{\mathcal{N}=8}^{(n)} = \int_{P(\mathbb{S}_{[8]}^+)} d^{4|16}x \prod_{i=1}^n \frac{[\pi_i d\pi_i]}{[\pi_{i+1} \pi_i]} \frac{\mathcal{H}_n}{[\pi_1 \pi_n]} \frac{\mathcal{H}_{n-1}}{[\pi_n \pi_{n-1}]} \mathcal{V}_{n-2} \cdots \mathcal{V}_2 \left(\frac{\mathcal{H}_1}{[\pi_{n-1} \pi_1]} \right). \quad (93)$$

Unlike the previous formulæ (68) & (88), this expression singles out *three* of the external fields, representing them in terms of the Hamiltonian function \mathcal{H} rather than the vector field \mathcal{V} . This is closely related to the formula obtained by Nair in [46]. It is easy to check that (93) reproduces the BGK amplitudes for external gravitons, and satisfies the supersymmetric recursion relations of [15].

7 Conclusions and future directions

A perspective of this paper has been that the MHV vertices provide a bridge between perturbative treatments of gravity and the fully nonlinear, non-perturbative structure that is such a key part of General Relativity. When we are on-shell with respect to the chiral action (81) (or the chiral limit of the Plebanski action), we may take advantage of the integrability of the anti self-dual Einstein equations to interpret the infinite sum of MHV amplitudes as simply the square of a linearized fluctuation γ of the self-dual spin connection Γ on the ASD background. The techniques of this paper, both for Yang-Mills and gravity indicate that it is possible to develop a background field formalism on fully nonlinear asd backgrounds within which explicit computations are tractable and generate amplitudes for processes with an arbitrary number of negative helicity legs. This programme would allow one to incorporate the integrability of the anti self-dual Einstein equations into the study of perturbation theory in such a way as to bridge the gap between perturbative and non-perturbative treatments of gravity.

The status of the MHV diagram formulation for gravity is currently less clear than that for Yang-Mills, although it has now been verified for up to 11 external particles [15]. At this stage there is no reason to doubt that the MHV picture for gravity should be successful, at least classically. The validity of our twistor action (85) for gravity currently depends on that of the MHV formalism whereas, in the case of Yang-Mills, the twistor action of [32, 33] and reviewed in appendix B provides an independent non-perturbative derivation of the MHV formalism [40]. A future goal is to construct a twistor action for gravity that works in the same way—for this it will be necessary to build a formalism in which the background is *off-shell* and \mathcal{PT} possesses only an *almost* complex structure. A search for a spacetime MHV Lagrangian for gravity has been initiated in [43], following the path of [44] in Yang-Mills.

$\mathcal{N} = 4$ supergravity is not unique, and (88) is not the unique $\mathcal{N} = 4$ completion of the non-supersymmetric action. Firstly, the Poisson structure I may also point along the fermionic directions, and in [38, 65] this was shown to be responsible for gauged supergravities in the self-dual sector. Secondly, unlike the $\mathcal{N} = 4$ completion of the MHV amplitudes in Yang-Mills, there seems to be no compelling reason that the nonlocal term in (88) should be only *quadratic* in \mathcal{B} . It would be interesting to know if additional terms are required in the case of gauged supergravity.

A key motivation for much of the work here is to reverse engineer a twistor-string theory for gravity. The Lie derivatives and inverse $\bar{\partial}$ -operators in the second term in the action (88) are suggestive of a worldsheet OPE interpretation, and it would be fascinating to see if this term (taken on-shell) can arise as an instanton contribution in some form of twistor-string theory. In particular, h and B enter just as they do in the vertex operators of [36].

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A Simplifying the BGK Amplitudes

In this appendix we will show analytically that the Berends, Giele & Kuijf [18] form of the graviton MHV amplitude agrees with the simplified expression (1) used in the text. Similar manipulations have been performed in [5, 16, 46]; our version of the amplitude is nearest to one given implicitly in [46], although we believe the detailed form is new.

Berends, Giele & Kuijf give the MHV amplitude

$$\mathcal{M}_{\text{BGK}} = \frac{\kappa^{n-2}}{\hbar} \delta^{(4)} \left(\sum p \right) M, \quad (94)$$

where for $n \geq 5$

$$M(1^+, 2^-, 3^-, \dots, n-1^-, n^+) = [1n]^8 \left\{ \frac{\langle 12 \rangle \langle n-2 \ n-1 \rangle}{[1 \ n-1]} \frac{F}{N(n)} \prod_{i=1}^{n-3} \prod_{j=i+2}^{n-1} [ij] + P_{(2, \dots, n-2)} \right\} \quad (95)$$

with $N(n) := \prod_{i < j} [ij]$ and where

$$F := \prod_{k=3}^{n-3} \langle k | p_{k+1} + p_{k+2} + \dots + p_{n-1} | n \rangle \quad (96)$$

when $n \geq 6$ and $F = 1$ when $n = 5$. In (95), the symbol $P_{2, \dots, n-2}$ denotes a sum over all permutations of gravitons 2 to $n-2$.

We begin by writing

$$\begin{aligned} \frac{\langle 12 \rangle \langle n-2 \ n-1 \rangle}{[1 \ n-1]} &= \frac{\langle 21 \rangle [1n] \langle n-2 \ n-1 \rangle [n-1 \ n]}{[1 \ n-1] [n-1 \ n] [n1]} \\ &= - \frac{\langle 2 | p_3 + p_4 + \dots + p_{n-1} | n \rangle \langle n-2 | p_{n-1} | n \rangle}{[1 \ n-1] [n-1 \ n] [n1]} \end{aligned} \quad (97)$$

using momentum conservation in the second step. Combining this with F in equation (96) gives a factor

$$- \frac{\prod_{k=2}^{n-2} \langle k | p_{k+1} + p_{k+2} + \cdots + p_{n-1} | n \rangle}{[1 \ n - 1][n - 1 \ n][n 1]} \quad (98)$$

Next, by carefully altering the limits of the products, we may re-express $N(n)$ as

$$N(n) = \prod_{i=1}^{n-1} \prod_{j=i+1}^n [ij] = -C(n) \left\{ \prod_{i=1}^{n-3} \prod_{j=i+2}^{n-1} [ij] \right\} \prod_{k=2}^{n-2} [kn] \quad (99)$$

where $C(n)$ is the cyclic product $[12][23] \cdots [n-1 \ n][n 1]$. The term in braces now cancels an identical term in the numerator of (95). Hence we obtain

$$\mathcal{M}_{\text{BGK}}(1^+, 2^-, 3^-, \dots, n-1^-, n^+) = \frac{\kappa^2}{\hbar} \delta \left(\sum p \right) \times \left\{ \frac{[1n]^8}{[1 \ n - 1][n - 1 \ n][n 1]} \frac{1}{C(n)} \prod_{k=2}^{n-2} \frac{\langle k | p_{k+1} + \cdots + p_{n-1} | n \rangle}{[kn]} + P_{(2, \dots, n-2)} \right\}, \quad (100)$$

which is the form of the amplitudes in equation (1).

B Yang-Mills

In this appendix, we will review the twistor construction of the Parke-Taylor amplitudes in Yang-Mills theory (see [32, 33] for further details). Although this section is not strictly necessary for an understanding of the gravitational case, there are nonetheless many analogies between the two and some readers may find it useful to refer here for comparison.

B.1 Scattering off an Anti Self-Dual Yang-Mills Background

On spacetime, Yang-Mills theory may be described by the Chalmers & Siegel [66] action

$$S[A, G^+] = \frac{1}{g^2} \int_M \text{tr} (G^+ \wedge F - G^+ \wedge G^+), \quad (101)$$

where $F = dA + A^2$ and G^+ is a Lie algebra-valued self-dual 2-form. We will frequently drop the superscript from G^+ , but it is always self-dual. The field equations are

$$G^+ = \frac{1}{2} F^+ \quad \text{and} \quad D_A G^+ = 0, \quad (102)$$

where D_A is the covariant derivative. The first of these equations may be viewed as a constraint; enforcing it in (101) one recovers the standard Yang-Mills action, upto a topological term. Using the Bianchi identity, the second equation is the standard Yang-Mills equations $D_A^* F = 0$.

Anti self-dual solutions to (102) have $F^+ = 0$. Replacing $A \rightarrow A + a$ and $G \rightarrow G + g$ and expanding the full field equations to linear order, one finds

$$2g = (D_A a)^+ \quad \text{and} \quad D_A g = 0 \quad (103)$$

when the background is anti self-dual. The solution space of these linear equations is an (infinite dimensional) vector space U (to be considered modulo gauge transformations). If \mathcal{R} denotes the space of solutions to the full equations (102), then U may be interpreted as the fibre of $T\mathcal{R}$ over a particular ASD solution. As for gravity, we identify $U^- \subset U$ as the subspace with $g = 0$. Since $F \rightarrow F + D_A a = F + (D_A a)^+ + (D_A a)^-$, equation (103) shows that linearized solutions in U^- preserve the anti self-duality of the Yang-Mills curvature. U^+ is defined asymmetrically to be $U^+ := \{g \in \Omega^{2+}(M, \text{End} E) \mid D_A g = 0\}$, modulo gauge transformations. From equation (103), such g fields generate linear fluctuations in the self-dual part of the curvature. However, on an ASD background it does not make sense to ask for U^+ to be the solutions that are *purely* self-dual, because under a background gauge transformation with parameter χ , the variation $a \rightarrow a + D_A \chi$ implies

$$D_A a \rightarrow D_A a + D_A(D_A \chi) = D_A a + [F^-, \chi] \quad (104)$$

so that requiring $(D_A a)^- = 0$ would not be invariant under background gauge transformations. Again, this is summarized by the exact sequence

$$0 \rightarrow U^- \rightarrow U \rightarrow U^+ \rightarrow 0 \quad (105)$$

where $U^- \rightarrow U$ is an inclusion and the map $U \rightarrow U^+$ is $(a, g) \mapsto g$. The fact that a self-dual fluctuation may or may not have an anti self-dual component obstructs the global splitting $U = U^- \oplus U^+$. Once again, this obstruction may be attributed to the MHV amplitudes, interpreted as scattering a linearized self-dual field off the ASD background.

Evaluating the action (101) on $(A, G^+) = (A_0 + a, g)$ where $(A_0, 0)$ are an ASD background and (a, g) obey the linearized equations (103), we find

$$\frac{i}{\hbar} S[A_0 + a, g] = \frac{i}{g^2 \hbar} \int_M \text{tr}(g \wedge g) \quad (106)$$

which, according to the path-integral argument in section 2, is the tree-level amplitude for a positive helicity gluon to scatter off the background and emerge with negative helicity. We can again confirm this with a separate calculation.

The space of solutions \mathcal{R} of (102) again possesses a naturally defined closed two-form

$$\Omega := \frac{1}{g^2} \int_C \text{tr}(\delta G \wedge \delta A) . \quad (107)$$

As a consequence of the field equations, Ω is independent of the Cauchy surface C and descends to a symplectic form on \mathcal{Y}/gauge . If $\mathcal{A}_{1,2} = (a_{1,2}, g_{1,2})$ are two sets of linearized solutions, then $g^2 \Omega(\mathcal{A}_1, \mathcal{A}_2) = \int_C \text{tr}(g_1 \wedge a_2 - g_2 \wedge a_1)$. As for gravity on an anti self-dual spacetime, the symplectic form (107) can be used to define a splitting of U that depends on a choice of Cauchy surface C . Clearly, U^- forms a Lagrangian subspace with respect to (107) and we can ensure U^+ is likewise Lagrangian by *defining* a fluctuation \mathcal{A}_2 to be purely self-dual if

$$\Omega(\mathcal{A}_2, \mathcal{A}_1) = -\frac{1}{g^2} \int_C \text{tr}(g_2 \wedge a_1) \quad (108)$$

for an *arbitrary* fluctuation \mathcal{A}_1 .

The quantum mechanical inner-product is defined in the same way as in the text (on Minkowski space \mathbb{M} the positive/negative frequency splitting can be performed straightforwardly) and agrees with the symplectic form (107) on positive frequency states. The amplitude for a linearized fluctuation \mathcal{A}_1 that has positive helicity and positive frequency at \mathcal{I}^- to emerge at \mathcal{I}^+ with *negative* helicity (and positive energy) after traversing region of anti self-dual Yang-Mills curvature (in \mathbb{M}) is $\langle \mathcal{A}_2 | \mathcal{A}_1 \rangle_{\text{asd}}$, where \mathcal{A}_2 is purely self-dual at \mathcal{I}^+ . In exact analogy to equation (21), we find

$$\begin{aligned} \langle \mathcal{A}_2 | \mathcal{A}_1 \rangle_{\text{asd}} &= \frac{i}{g^2 \hbar} \int_{\mathcal{I}^+} \text{tr} (g_2 \wedge a_1) \\ &= \frac{i}{g^2 \hbar} \int_{\mathbb{M}} \text{tr} (D_A g_2 \wedge a_1 + g_2 \wedge D_A a_1) + \frac{i}{g^2 \hbar} \int_{\mathcal{I}^-} \text{tr} (g_2 \wedge a_1) \\ &= \frac{i}{g^2 \hbar} \int_{\mathbb{M}} \text{tr} (g_2 \wedge g_1) \end{aligned} \quad (109)$$

after using the linearized field equations (103) and the fact that \mathcal{A}_1 is purely self-dual at \mathcal{I}^- .

Equation (109) is a generating function for the Parke-Taylor amplitudes. To obtain them in their usual form, one must construct a background ASD field A that is a (non-linear) superposition of $n - 2$ plane waves and solve the equation $D_A g = 0$ with such an A . Finally, one must expand the above integral to the appropriate order. As for gravity, these problems are considerably simplified by the use of twistor theory, which brings out the integrability of the ASD Yang-Mills equations.

B.2 The Twistor Theory of Yang-Mills

For the basic notation of twistor space, we refer to the beginning of section 3. Anti self-dual connections on spacetime correspond to holomorphic bundles E on twistor space, by the Ward construction [67]. In the Dolbeault framework used in this paper, such a bundle is determined by an operator $\bar{\partial} + \mathcal{A}$ satisfying $\mathcal{F}^{(0,2)} := (\bar{\partial} + \mathcal{A})^2 = 0$, where $\bar{\partial}$ is the standard $\bar{\partial}$ -operator on twistor space and \mathcal{A} is the $(0, 1)$ -form part of a connection on E (and has homogeneity degree 0). Note that $\bar{\partial} + \mathcal{A}$ may be regarded as a deformation of the $\bar{\partial}$ -operator on a flat gauge bundle, while the integrability condition $\mathcal{F}^{(0,2)} = 0$ arises as the field equations of the action

$$\int_{\mathbb{P}\mathbb{T}'} \Omega \wedge \text{tr}(\mathcal{G} \wedge \mathcal{F}) \quad (110)$$

where \mathcal{G} is a $(0, 1)$ -form of homogeneity -4 with values in $\text{End}(E)$ and Ω is the canonical holomorphic $(3, 0)$ -form of weight $+4$ on $\mathbb{P}\mathbb{T}'$. Thus, (110) is the twistor equivalent of the $g^2 \rightarrow 0$ limit of (101) on spacetime.

Following Sparling [68], the spacetime Yang-Mills connection can be reconstructed by first solving

$$(\bar{\partial} + \mathcal{A})|_{L_x} H = 0, \quad (111)$$

where, for a Yang-Mills field on spacetime with gauge group G , \mathcal{A} takes values in the complexified Lie algebra of G whereas H is valued in the complexification of G itself.

The notation $(\bar{\partial} + \mathcal{A})|_{L_x}$ means the restriction of the twistor space operator $\bar{\partial} + \mathcal{A}$ to L_x . A solution H of (111) is a global holomorphic frame of $E|_{L_x}$, related to the twistor connection one-form by

$$\mathcal{A}|_{L_x} = -\bar{\partial}H H^{-1} . \quad (112)$$

The generic existence of such frames for each x is guaranteed by standard properties of holomorphic vector bundles¹⁶. To reconstruct the spacetime connection A , first note that $H^{-1}\pi^{\dot{\alpha}}\partial H/\partial x^{\alpha\dot{\alpha}}$ has homogeneity one in $\pi_{\dot{\alpha}}$. Moreover, $H^{-1}\pi^{\dot{\alpha}}\partial H/\partial x^{\alpha\dot{\alpha}}$ is holomorphic on L_x , since

$$\bar{\partial} \left(H^{-1}\pi^{\dot{\alpha}} \frac{\partial H}{\partial x^{\alpha\dot{\alpha}}} \right) = H^{-1}\mathcal{A}\pi^{\dot{\alpha}} \frac{\partial H}{\partial x^{\alpha\dot{\alpha}}} - H^{-1}\pi^{\dot{\alpha}} \frac{\partial}{\partial x^{\alpha\dot{\alpha}}}(\mathcal{A}H) = 0 , \quad (113)$$

where $\pi^{\dot{\alpha}}\partial\mathcal{A}/\partial x_{\alpha\dot{\alpha}} = 0$ because \mathcal{A} has been pulled back from \mathbb{PT} and so depends on x only through the combination $x^{\alpha\dot{\alpha}}\pi_{\dot{\alpha}}$. Thus $H^{-1}\pi^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}H$ must in fact be linear in $\pi_{\dot{\alpha}}$ and so may be written as

$$H^{-1}\pi^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}H = \pi^{\dot{\alpha}}A_{\alpha\dot{\alpha}}(x) \quad (114)$$

for some Lie-algebra valued functions $A_{\alpha\dot{\alpha}}$ that depend only on spacetime. This provides the spacetime connection $A = A_{\alpha\dot{\alpha}}dx^{\alpha\dot{\alpha}}$.

To construct a twistor expression for $\langle \mathcal{A}_2 | \mathcal{A}_1 \rangle$, recall that for a flat Yang-Mills bundle, the Penrose transform of a linearized fluctuation g is related to \mathcal{G} by

$$g_{\dot{\alpha}\dot{\beta}}(x) = \int_{L_x} [\pi d\pi] \wedge \pi_{\dot{\alpha}}\pi_{\dot{\beta}} p^*(\mathcal{G}) \quad (115)$$

where $g = g_{\dot{\alpha}\dot{\beta}}dx^{\alpha\dot{\alpha}} \wedge dx_{\alpha}^{\dot{\beta}}$. Moreover, if $\bar{\partial}\mathcal{G} = 0$ then $g_{\dot{\alpha}\dot{\beta}}$ automatically obeys $\partial^{\alpha\dot{\alpha}}g_{\dot{\alpha}\dot{\beta}} = 0$, again because the pullback $p^*\mathcal{G}$ depends on x only through $x^{\alpha\dot{\alpha}}\pi_{\dot{\alpha}}$. The equations $\bar{\partial}\mathcal{G} = 0$ and $\partial^{\alpha\dot{\alpha}}G_{\dot{\alpha}\dot{\beta}} = 0$ are the linearized field equations of (110) and (101) in the case that the background bundles are flat¹⁷ so that we can find a gauge where $\mathcal{A} = 0$ and $A = 0$. However, on a ASD Yang-Mills background, (115) does not quite make sense. In order to add up an $\text{End}(E)$ -valued form over L_x , in the presence of a non-flat Yang-Mills bundle we need first to pick a holomorphic trivialization of $E|_{L_x}$ that is global over L_x : this is just the solution H of equation (111). The background-coupled twistor integral formula for g^+ is then

$$g_{\dot{\alpha}\dot{\beta}}(x) = \int [\pi d\pi] \wedge \pi_{\dot{\alpha}}\pi_{\dot{\beta}} H^{-1}(x, \pi) p^*(\mathcal{G}) H(x, \pi) . \quad (116)$$

From equation (116) we now find (dropping the pullback symbol p^*)

$$\begin{aligned} \partial^{\alpha\dot{\alpha}}g_{\dot{\alpha}\dot{\beta}} &= \int [\pi d\pi] \wedge \pi_{\dot{\alpha}}\pi_{\dot{\beta}} \frac{\partial}{\partial x^{\alpha\dot{\alpha}}} (H^{-1}\mathcal{G}H) \\ &= \int [\pi d\pi] \wedge \pi_{\dot{\alpha}}\pi_{\dot{\beta}} (-\pi_{\dot{\alpha}}A^{\alpha\dot{\alpha}}H^{-1}\mathcal{G}H + H^{-1}\mathcal{G}HA^{\alpha\dot{\alpha}}) = -[A^{\alpha\dot{\alpha}}, g_{\dot{\alpha}\dot{\beta}}] \end{aligned} \quad (117)$$

¹⁶The Penrose-Ward transform requires $E|_{L_x}$ to be trivial. This will generically be the case and arises because the fibre of the Yang-Mills bundle over a spacetime point x is by definition the space of global holomorphic sections of $E|_{L_x}$; these ‘jump’ if $E|_{L_x}$ becomes non-trivial, so any twistor bundle that comes from a spacetime bundle will necessarily be trivial over L_x .

¹⁷They can also be thought of as Abelianized versions of the full theory.

or in other words $D_A g = 0$, which is the linearized field equation (103) for g on an ASD background. Therefore, the scattering amplitude we seek is given by

$$\langle \mathcal{A}_2 | \mathcal{A}_1 \rangle_{\text{asd}} = \frac{i}{g^2 \hbar} \int d^4 x [\pi_1 d\pi_1] [\pi_2 d\pi_2] [\pi_1 \pi_2]^2 \text{tr} (H_2^{-1} \mathcal{G}_2 H_2 H_1^{-1} \mathcal{G}_1 H_1) \quad (118)$$

where the integral on the right is then taken over $\mathbb{R}^4 \times \mathbb{CP}^1 \times \mathbb{CP}^1$.

To obtain the Parke-Taylor amplitudes we must expand the frames H as a perturbation series around a flat background by inverting the relation $\mathcal{A}|_{L_x} = -\bar{\partial} H H^{-1}$. Rather than do this directly (see [47]), it is simpler to note that the Green's function K_{12} for the $\bar{\partial}$ -operator on L_x , acting on sections of $\text{End}(E)|_{L_x}$, is related to H by

$$K_{12}(x, \pi_1, \pi_2) = \left(\frac{1}{2\pi i} \right) \frac{H(x, \pi_1) H^{-1}(x, \pi_2)}{[\pi_1 \pi_2]} \quad (119)$$

and may formally be thought of as $(\bar{\partial} + \mathcal{A}|_{L_x})^{-1}$. This is analogous to equation (64) in section 4. The Green's function thus depends non-polynomially on \mathcal{A} ; expanding the right hand side of (118) as a series in \mathcal{A} using $K_{ij}|_{\mathcal{A}=0} = 1/2\pi i [\pi_i \pi_j]$, one obtains

$$\frac{i}{g^2 \hbar} \int d^4 x \prod_{i=1}^n \frac{[\pi_i d\pi_i]}{[\pi_i \pi_{i+1}]} \sum_{p=2}^n [\pi_1 \pi_p]^4 \text{tr} (\mathcal{A}_n \cdots \mathcal{A}_{p+1} \mathcal{G}_p \mathcal{A}_{p-1} \cdots \mathcal{A}_2 \mathcal{G}_1) \quad (120)$$

for the vertex involving n fields. To obtain the Parke-Taylor amplitudes, take \mathcal{A} and \mathcal{G} to be linear combinations of momentum eigenstates momentum $p_{\alpha\dot{\alpha}} = \tilde{k}_{\alpha} k_{\dot{\alpha}}$, with helicities -1 and $+1$, respectively. As in [57], these can be represented by the twistor functions,

$$\begin{aligned} \mathcal{A} &= g \sum_{i=3}^n \epsilon_i T_i \bar{\delta}_{(0)}([\pi i]) \exp \left(\frac{\langle \omega i \rangle [i \sigma]}{[\pi \sigma]} \right) \\ \mathcal{G} &= g \sum_{i=1}^2 \epsilon_i T_i \bar{\delta}_{(-4)}([\pi i]) \exp \left(\frac{\langle \omega i \rangle [i \sigma]}{[\pi \sigma]} \right), \end{aligned} \quad (121)$$

where T_i are arbitrary elements of the Lie algebra of the gauge group, and the ϵ_i are expansion parameters. The coefficient of $\prod_{i=1}^n \epsilon_i$ in (120) is the n^{th} -order Parke-Taylor amplitude (complete with the appropriate colour-trace).

Treating the fields \mathcal{A} and \mathcal{G} as $\text{End } E$ -valued $(0,1)$ -forms, rather than representatives of cohomology classes, we can combine (120) with the action (110) to obtain a twistor action for the MHV diagram formulation of Yang-Mills. It is straightforward to extend this to an action for $\mathcal{N} = 4$ SYM. See [32, 33, 40] for details.

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